Primordial Black Holes and Stochastic Inflation beyond slow roll
Cosmology from Home @2023

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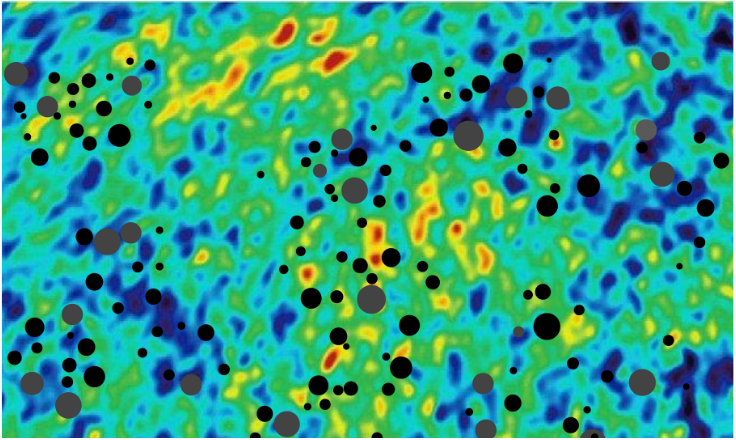
Primordial Black Holes (PBHs)

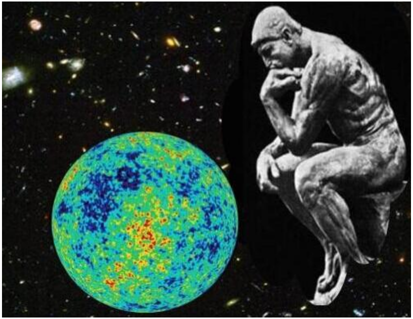


Candidates for Dark Matter, Hawking Radiation, Baryogenesis, Reheating, seeds of SMBHs etc.

Extremely interesting rich phenomenology!

Inflation, Quantum fluctuations and PBHs

CMB $\longrightarrow$ LSS



- **Adiabatic** $\zeta(\vec{x})$
- Almost **scale-invariant**

$$\boxed{\mathcal{P}_\zeta = A_S \left(\frac{k}{k_*}\right)^{n_S}}$$

$$A_S \simeq 2 \times 10^{-9}\,,\ \ n_{_S} \simeq -0.035$$

- Nearly **Gaussian**

$$\boxed{P[\zeta] = \mathcal{B} \exp\left[\frac{-\zeta^2}{2\sigma^2}\left(1 + f_{\rm NL}\,\zeta + ...\right)\right]}$$

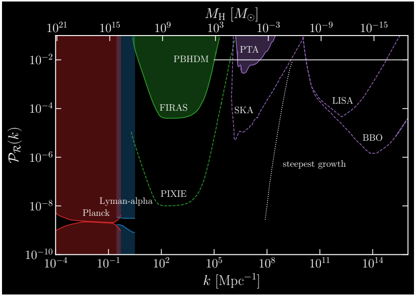
$\rightarrow$ **LSS, CMB $\Rightarrow$ Large-scale tiny quantum fluctuations**

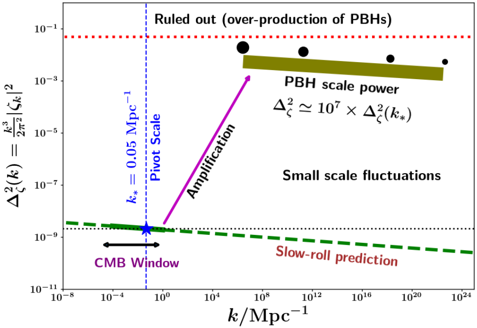
$\rightarrow$ **PBHs, $\mathrm{GW}^{(2)}$s $\Rightarrow$ Small-scale larger fluctuations ?**

What we know from Observations

CMB probes scales $k \in [0.0005,\ 0.5]\ \ \mathrm{Mpc}^{-1} \Rightarrow \Delta N \simeq 7$

Small-scale power spectrum is not constrained!



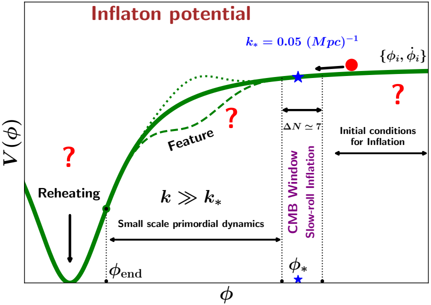


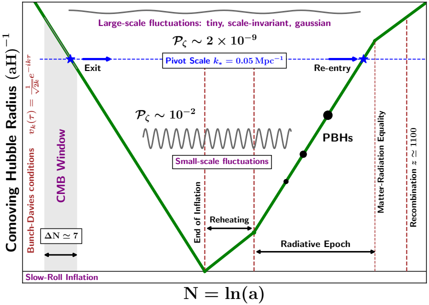
Possibility of enhancement of small-scale fluctuations!

**Green and Kavanagh, J. Phys. G 48 (2021) 4, 043001

Single-field Inflation beyond the CMB Window

⇒ **Scope for non-trivial small-scale dynamics**





$$\textbf{CMB scales}: \ P_\zeta \sim k^{-0.035} \quad (\text{Slightly red} - \text{tilted})\ ; \quad \eta_H \simeq -0.018$$

$$\textbf{Small-scale growth}: \ P_\zeta \sim k^{n_S\ (\leq 4)} \quad (\text{Blue} - \text{tilted})\ ; \quad \eta_H \geq 3/2$$

**Byrnes et. al JCAP 06(2019) 028

Large Quantum Fluctuations

1. **Breakdown of scale-invariance at small-scales**

$$\boxed{\epsilon_H = -\frac{\mathrm{d}\ln H}{\mathrm{d}N}\,, \quad \eta_H = \epsilon_H - \frac{1}{2}\frac{\mathrm{d}\ln\epsilon_H}{\mathrm{d}N}} \quad ; \quad \mathrm{N} = \ln(\mathrm{a})$$

2. **Breakdown of Gaussian nature of primordial fluctuations**

For $\zeta \gg 1$

$$\boxed{P[\zeta] \neq \mathcal{B}\exp\left[\frac{-\zeta^2}{2\int_{k_1}^{k_2}\mathrm{d}\ln k\,\mathcal{P}_\zeta(k)}\left(1 + f_{\mathrm{NL}}\,\zeta + g_{\mathrm{NL}}\,\zeta^2 + \ldots\right)\right]}$$

****Celoria et. al JCAP 06 (2021) 051**

Breakdown of Scale-invariance via feature

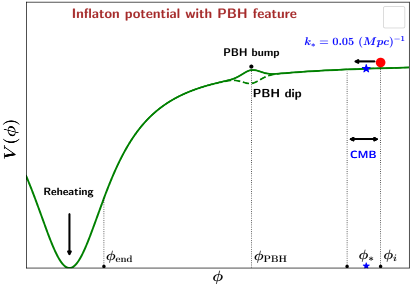
A feature: **an inflection point** or **a local bump/dip** at **low scales** slows down the inflaton

⇒ **Breaking of scale invariance!!**

At small scales $\boxed{\epsilon_H \ll 1\,, \quad \eta_H \gtrsim 3}$ **Violation of slow-roll**

Criteria for PBH from single field Inflation–

1. Large scales satisfying with CMB constraints.
2. Intermediate scale feature to enhance power for PBH formation.
3. Successful Reheating mechanism.



**Motohashi, Hu PRD 96(2017) 6, Cole et. al arXiv:2304.01997

Computing Power spectrum

$$\boxed{\mathcal{P}_\zeta(k) = \frac{k^3}{2\pi^2}\,|\zeta_k|^2}_{k<<aH}$$

Mukhanov-Sasaki variable $\boxed{v_k = z \times \zeta_k}$; $z = a m_p \sqrt{2\epsilon_H}$

In **spatially-flat gauge**

$$\boxed{\frac{\mathrm{d}^2 v_k}{\mathrm{d}N^2} + (1-\epsilon_H)\frac{\mathrm{d}v_k}{\mathrm{d}N} + \left[\left(\frac{k}{aH}\right)^2 + M_{\text{eff}}^2(N)\right] v_k = 0}$$

where the **effective mass term** is

$$\boxed{M_{\text{eff}}^2(N) = -\frac{1}{(aH)^2}\left[2 + 2\epsilon_H - 3\eta_H + 2\epsilon_H^2 + \eta_H^2 - 3\epsilon_H\eta_H - \frac{\mathrm{d}\eta_H}{\mathrm{d}N}\right]}$$

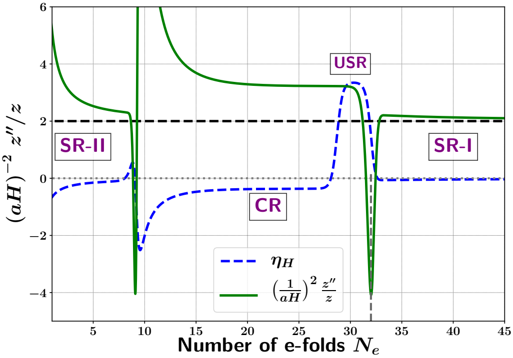
Background dynamics dependent and complicated

Typical Inflationary Dynamics

SR-I (CMB scale) $\longrightarrow$ **USR** $\longrightarrow$ **CR** $\longrightarrow$ **SR-II**

$$\boxed{\eta_H : \quad \eta_1 \longrightarrow \eta_2 \longrightarrow \eta_3 \longrightarrow \eta_4}$$ **Wands Duality**

Background



Reason for duality

For $\epsilon_H \ll 1$,

$$\frac{M_{\rm eff}^2}{(aH)^2} \simeq 2 - 3\eta_H + \eta_H^2 - \frac{{\rm d}\eta_H}{{\rm d}N}$$

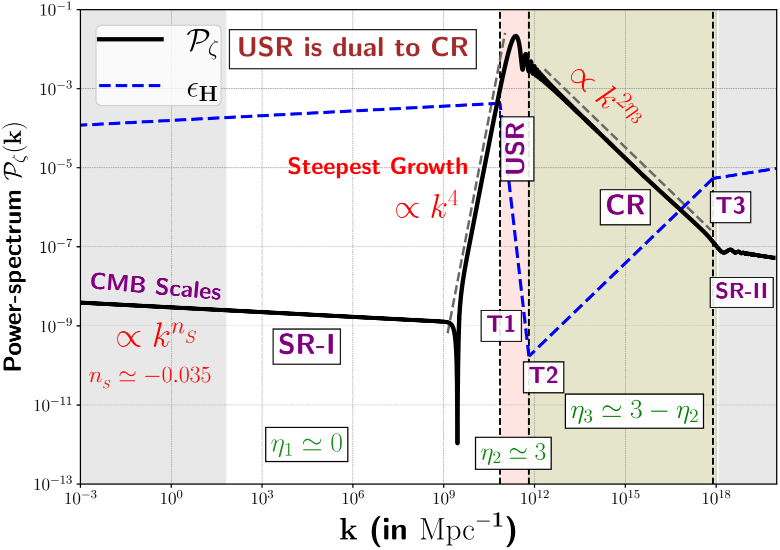
Assuming

$$\boxed{\eta_H = \frac{3}{2} + C\, \tanh\left[C\left(N_e - \tilde{N}_e\right)\right]}$$

$$\nu^2 \equiv \frac{M_{\rm eff}^2}{(aH)^2} + \frac{1}{4} \simeq \text{const.}$$

****SSM, Sahni JCAP 04(2020) 007,** ****Karam et. al JCAP 03(2023) 013**

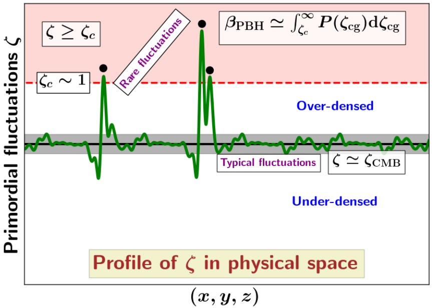
Typical Power-spectrum

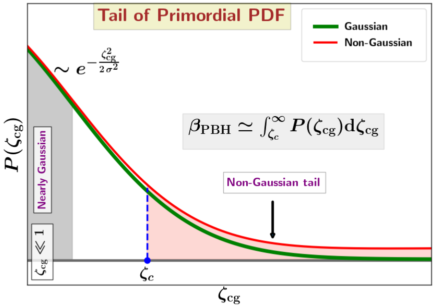


Statistics of Primordial Fluctuations

Is the PDF of Primordial Fluctuations $P[\zeta]$ Gaussian or Non-Gaussian?

Non-Gaussian for $\zeta \gg 1$ in general





PBHs from Rare Peaks: Sensitive to the tail of PDF

Non-Perturbative Methods for full PDF

Approach - I

Classical Non-linear δN formalism

Approach - II

Semi-classical Approximation

Approach - III

Stochastic Inflation

Stochastic Inflation: Effective IR description

Coarse-grained description $\phi = \Phi + \varphi \ , \ \pi_\phi = \Pi + \pi$

Langevin Equations (Non-linear)

$$\frac{\mathrm{d}\Phi}{\mathrm{d}N} = D_\Phi + \xi_\phi \, ; \quad \frac{\mathrm{d}\Pi}{\mathrm{d}N} = D_\Pi + \xi_\pi$$

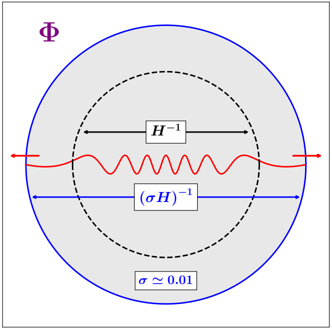
$$\frac{\mathrm{d}F_{\rm cg}}{\mathrm{d}N} = \textbf{Drift}_{\rm cl} + \textbf{Diffusion}_{\rm Q}$$

Gaussian White noise statistics

$$\langle \xi_i(N)\, \xi_j(N') \rangle = \Sigma_{ij}(N)\, \delta_D(N - N')$$

Noise Matrix elements

$$\Sigma_{ij}(N) = (1 - \epsilon_H)\frac{k^3}{2\pi^2}\phi_{i_k}(N)\phi^*_{j_k}(N)\Big|_{k=\sigma a H}$$



Coarse-graining scale

$$k = \sigma\, aH \, , \quad \sigma \ll 1$$

****A. A. Starobinsky (1986)**

PDF from first-passage time analysis

$$\frac{\mathrm{d}\Phi}{\mathrm{d}N} = D_\Phi + \xi_\phi \,; \qquad \frac{\mathrm{d}\Pi}{\mathrm{d}N} = D_\Pi + \xi_\pi$$

First-passage no. of e-folds $\mathcal{N}$

and **PDF** $P(\mathcal{N})$

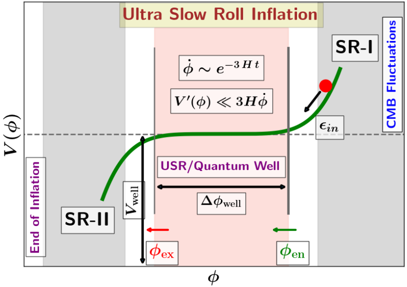
Subject to boundary conditions

1. **Reflecting boundary** at $\Phi = \phi_{\rm en}$:

$$\frac{\partial}{\partial \Phi} P(\mathcal{N})\Big|_{\Phi=\phi_{\rm en}} = 0$$

2. **Absorbing boundary** at $\Phi = \phi_{\rm ex}$:

$$P(\mathcal{N})\Big|_{\Phi=\phi_{\rm ex}} = \delta_D(\mathcal{N})$$



- Numerical Simulations
- **Fokker-Planck Equation** (suited for analytical treatment)

Langevin $\longrightarrow$ Fokker-Planck Equation

PDF of first-passage number of e-foldings $\mathcal{N}$: **Adjoint FPE**

$$\frac{\partial P(\mathcal{N})}{\partial \mathcal{N}} = \left[D_\Phi \frac{\partial}{\partial \Phi} + D_\Pi \frac{\partial}{\partial \Pi} + \frac{1}{2}\Sigma_{\phi\phi} \frac{\partial^2}{\partial \Phi^2} + \Sigma_{\phi\pi} \frac{\partial^2}{\partial \Phi \partial \Pi} + \frac{1}{2} \Sigma_{\pi\pi} \frac{\partial^2}{\partial \Pi^2} \right] P(\mathcal{N})$$

$$P(\mathcal{N}) \equiv P_{\Phi,\Pi}(\mathcal{N})$$

Stochastic $\delta\mathcal{N}$ Formalism

Statistics of $\mathcal{N}$ → Statistics of $\zeta_{\rm cg}$: $P[\mathcal{N}] \longrightarrow P[\zeta_{\rm cg}]$

$$\boxed{\zeta_{\rm cg} \equiv \zeta(\Phi) = \mathcal{N} - \langle \mathcal{N}(\Phi) \rangle}\,; \quad \langle \mathcal{N}(\Phi) \rangle = \int_0^\infty \mathcal{N}\, P(\mathcal{N})\, \mathrm{d}\mathcal{N}$$

Abundance of PBHs $\boxed{\beta \sim \int_{\zeta_c}^{\infty} P(\zeta_{\rm cg})\, \mathrm{d}\zeta_{\rm cg}}$

****Pattison et. al JCAP 04 (2021) 080**

Quasi de Sitter approximation

Mode functions $\{\phi_k, \pi_k\} \longrightarrow \mathrm{dS}$

$$\boxed{\Sigma_{\phi\phi} \simeq \left(\frac{H}{2\pi}\right)^2 , \quad \Sigma_{\phi\pi}, \Sigma_{\pi\pi} \ll \Sigma_{\phi\phi}}$$

The Langevin equations become

$$\boxed{\frac{\mathrm{d}\Phi}{\mathrm{d}N} = D_\Phi + \frac{H}{2\pi}\,\xi\,; \qquad \frac{\mathrm{d}\Pi}{\mathrm{d}N} = D_\Pi}$$

with single **Gaussian white noise** ξ satisfying

$$\langle \xi(N) \rangle = 0\ , \quad \text{and} \quad \langle \xi(N)\xi(N') \rangle = \delta_D\left(N - N'\right)$$

Adj. Fokker-Planck Equation becomes

$$\frac{\partial P(\mathcal{N})}{\partial \mathcal{N}} = \left[\frac{H^2}{8\pi^2}\frac{\partial^2}{\partial \Phi^2} + D_\Phi \frac{\partial}{\partial \Phi} + D_\Pi \frac{\partial}{\partial \Pi}\right] P(\mathcal{N})$$

PDF for flat Quantum Well: Pure diffusion

$$V(\Phi) = V_0\,, \quad H^2 \simeq \frac{V_0}{3m_p^2}$$

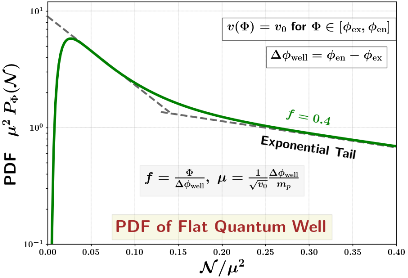
Leading to

PDF $$P(\mathcal{N}) = \sum_{n=0}^{\infty} A_n(\Phi)\, \mathrm{e}^{-\Lambda_n \mathcal{N}}$$

with $$\Lambda_n = (2n+1)^2 \frac{\pi^2}{4} \frac{1}{\mu^2}$$

$$A_n = (2n+1)\frac{\pi}{\mu^2} \sin\left[(2n+1)\frac{\pi}{2}\left(\frac{\Phi}{\Delta\Phi}\right)\right]$$

Control Parameter : $$\mu = 2\sqrt{2}\pi \frac{\Delta\phi_{\rm well}}{H}$$



Exponential Tail

Highly Non-Gaussian!!

****Pattison et. al JCAP 10(2017) 046; Ezquiaga et. al. JCAP 03(2020) 029**

Additional Complications

- General form of the feature

$$V(\Phi) = V_0 + \frac{1}{2} m^2 \Phi^2 \pm \frac{\mu}{2} \Phi^3 + \frac{\lambda}{4} \Phi^4 \pm ...$$

- When inflaton **drift** is included

$$\frac{\partial}{\partial \mathcal{N}} P(\mathcal{N}) = \left[\frac{\Sigma_{\phi\phi}}{2} \frac{\partial^2}{\partial \Phi^2} + \left(D_\Phi \frac{\partial}{\partial \Phi} + D_\Pi \frac{\partial}{\partial \Pi} \right) \right] P(\mathcal{N})$$

- **Beyond the de Sitter mode functions for noise**

$$\frac{\partial P}{\partial \mathcal{N}} = \left[D_\Phi \frac{\partial}{\partial \Phi} + D_\Pi \frac{\partial}{\partial \Pi} + \frac{\Sigma_{\phi\phi}}{2} \frac{\partial^2}{\partial \Phi^2} + \Sigma_{\phi\pi} \frac{\partial^2}{\partial \Phi \partial \Pi} + \frac{\Sigma_{\pi\pi}}{2} \frac{\partial^2}{\partial \Pi^2} \right] P(\mathcal{N})$$

Recently concluded work

SSM, Edmund J. Copeland and Anne M. Green,

“*Primordial black holes and stochastic inflation beyond slow roll: I - Noise Matrix Elements*”

[arXiv:2303:17375]

Computing Noise Matrix Elements

$$\Sigma_{ij}(N) = (1-\epsilon_H)\frac{k^3}{2\pi^2}\phi^*_{i_k}(N)\phi_{j_k}(N)\Big|_{k=\sigma aH}\,; \qquad \phi_{i_k} \equiv \{\phi_k, \pi_k\}$$

$$\phi_k(N) = \frac{v_k(N)}{a}\,, \quad \pi_k(N) = \frac{\mathrm{d}\phi_k}{\mathrm{d}N}$$

Mukhanov-Sasaki variable v_k in **spatially-flat gauge**

$$\boxed{\frac{\mathrm{d}^2 v_k}{\mathrm{d}N^2} + (1-\epsilon_H)\frac{\mathrm{d}v_k}{\mathrm{d}N} + \left[\left(\frac{k}{aH}\right)^2 + M_{\rm eff}^2\right] v_k = 0}$$

where the **effective mass term** is

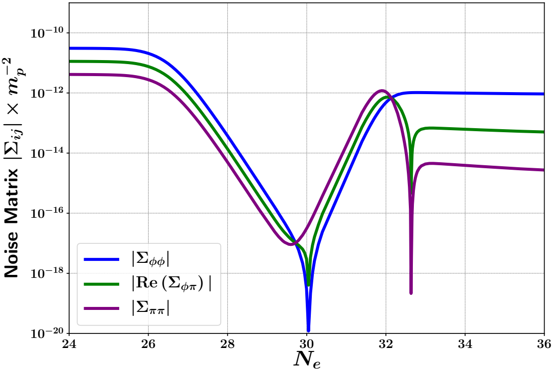
$$\boxed{-M_{\rm eff}^2 (aH)^{-2} = 2 + 2\epsilon_H - 3\eta_H + 2\epsilon_H^2 + \eta_H^2 - 3\epsilon_H\eta_H - \frac{\mathrm{d}\eta_H}{\mathrm{d}N}}$$

Background dynamics dependent and complicated

Numerical Noise Matrix Elements

Potential with a **tiny Gaussian bump/dip feature**

$$V(\phi) = V_0 \frac{\phi^2}{\phi^2 + M^2} \left[1 \pm A \, \exp\left(-\frac{1}{2} \left(\frac{\phi - \phi_0}{\Delta\phi} \right)^2 \right) \right]$$



Σ_{ij} evolves and swaps hierarchy! ****Mishra et. al JCAP 04(2020) 007**

Analytical appprox: Sharp transitions

Assume $|\epsilon_H| \ll |\eta_H|$ and $\epsilon_H \ll 1$ (**qdS** approx.)

$$\Rightarrow \boxed{\frac{z''}{z}(aH)^{-2} \simeq 2 - 3\eta_H + \eta_H^2 - \frac{1}{aH}\,\eta_H'}$$

And $\eta_H \to$ combination of **Step functions**

$$\boxed{\eta_H(\tau) = \eta_1 + (\eta_2 - \eta_1)\ \Theta(\tau - \tau_1) + ...}$$

For which

$$\boxed{\frac{z''}{z}(aH)^{-2} \simeq \mathcal{A}\,\tau\,\delta_D(\tau - \tau_1) + \left(\nu_1^2 - \frac{1}{4}\right) + \left(\nu_2^2 - \nu_1^2\right)\ \Theta(\tau - \tau_1) + ...}$$

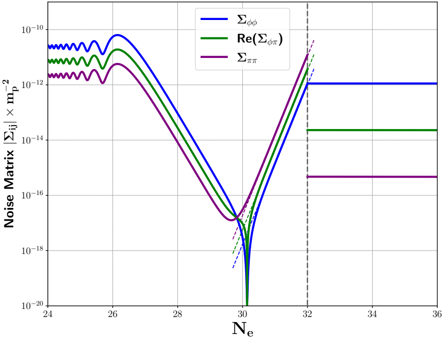
Where the **strength of transition** is $\boxed{\mathcal{A} = \eta_2 - \eta_1}$ and

order of Hankel $\boxed{\nu_{1,2}^2 = \left(\frac{3}{2} - \eta_{1,2}\right)^2}$

Results from Analytical Techniques

$$\boxed{\eta_H(\tau) = \eta_1 + (\eta_2 - \eta_1)\ \Theta(\tau - \tau_1)}\,, \qquad \text{Conformal time } \tau = \frac{-1}{aH}$$

$$\eta_1 \simeq -0.02\,; \qquad \eta_2 \simeq 3.3$$



Reproduces numerical results

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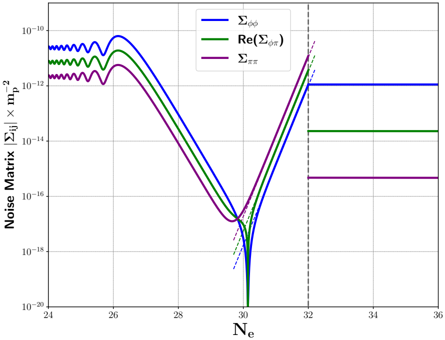

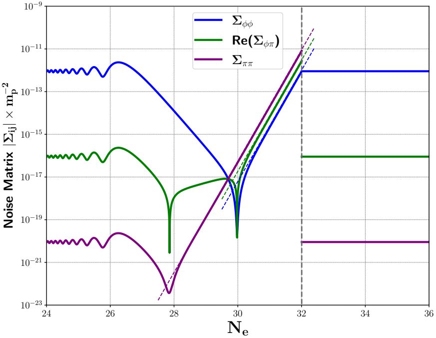

Reproduces numerical results **dS approximation**

Primary Conclusions

1. During **SR-I** phase, $\Sigma^{\rm SR}_{\phi\phi} \simeq \left(\frac{H}{2\pi}\right)^2$

$$\boxed{\Sigma_{\phi\phi} : |\Sigma_{\phi\pi}| : \Sigma_{\pi\pi} \simeq 1 : \left|\nu_1 - \frac{3}{2}\right| : \left(\nu_1 - \frac{3}{2}\right)^2}$$

2. Immediately after the transition, $\Sigma_{ij} \propto e^{-2\mathcal{A}N}$, and

$$\boxed{\Sigma_{\phi\phi} : |\Sigma_{\phi\pi}| : \Sigma_{\pi\pi} \simeq 1 : \mathcal{A} : \mathcal{A}^2}$$

3. During **CR** phase, $\Sigma^{\rm CR}_{\phi\phi} \simeq 2^{2(\nu_2-\nu_1)} \left[\frac{\Gamma(\nu_2)}{\Gamma(\nu_1)}\right]^2 \sigma^{2(\nu_1-\nu_2)} \Sigma^{\rm SR}_{\phi\phi}$

$$\boxed{\Sigma_{\phi\phi} : |\Sigma_{\phi\pi}| : \Sigma_{\pi\pi} \simeq 1 : \left|\nu_2 - \frac{3}{2}\right| : \left(\nu_2 - \frac{3}{2}\right)^2}$$

$\Rightarrow$ **Strongest diffusion during Constant-Roll epoch!**

Primary Conclusions

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$\Rightarrow$ **Strongest diffusion during Constant-Roll epoch!**

What is the nature of PDF $P[\zeta]$? Work in Progress

Caveats

1. Mode functions evolved in a fixed (deterministic background). **Figueroa *et. al* 2021
2. Computed in spatially-flat gauge. **Pattison *et. al* 2019
3. Only a single transition was considered analytically (duality).
4. Both Φ and Π were treated stochastically. **Tomberg 2022
5. β_{PBH} in terms of ζ rather than δ. **Tada, Vennin 2020

Questions (even basic ones) & Comments are most welcome.

Cosmology from Home is my favourite Conference!!

EXTRA SLIDES

Set-up for Analytical Computation

- $\eta_H \rightarrow$ is piece-wise constant $\Rightarrow \eta_i \simeq$ const.
- ν_i is piecewise (positive) constant.
- Introduce new time variable $\boxed{T = -k\tau = \dfrac{k}{aH}}$

$$\textbf{MS Eqn} \;\Rightarrow \boxed{\frac{\mathrm{d}^2 v_k}{\mathrm{d}T^2} + \left[1 - \frac{\nu^2 - 1/4}{T^2}\right] v_k = 0}$$

General solution is given by

1. **dS** mode functions

$$v_k(T) = \frac{1}{\sqrt{2k}} \left[\alpha_k \left(1 + \frac{i}{T}\right) e^{i\,T} + \beta_k \left(1 - \frac{i}{T}\right) e^{-i\,T}\right]$$

2. Beyond dS approximation

$$v_k(T) = \sqrt{T} \left[C_1\, H_\nu^{(1)}(T) \; + C_2\, H_\nu^{(2)}(T), \right]$$

Determining Co-efficients

- Apply Bunch-Davies initial conditions for modes exiting before the transition $T > T_1$

$$v_k(T)\Big|_{T\to\infty} \to \frac{1}{\sqrt{2k}}\, e^{iT}$$

- Apply **Israel Junction matching** conditions at transition

$$v_k^A(\tau_1) = v_k^B(\tau_1) \quad \text{(Continuity)}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} v_k^A\Big|_{\tau_1^+} - \frac{\mathrm{d}}{\mathrm{d}\tau} v_k^B\Big|_{\tau_1^-} = \int_{\tau_1^-}^{\tau_1^+} \mathrm{d}\tau \frac{z''}{z}\, v_k^A(\tau) \quad \text{(Differentiability)}$$

Application of the technique

- A **single instantaneous transition** from SR $\rightarrow$ USR using **dS** mode functions.
- A **single instantaneous transition** from SR $\rightarrow$ USR using Hankel functions.

Application of the technique

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Why a single transition ?

Wands duality between USR and CR

In the absence of transition

For $\nu =$ constant

$$\Sigma_{\phi\phi} = 2^{2\left(\nu-\frac{3}{2}\right)}\left[\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2\left(\frac{H}{2\pi}\right)^2 T^{2\left(\frac{3}{2}-\nu\right)}\left[1+\frac{1}{2(-1+\nu)}T^2+\ldots\right]$$

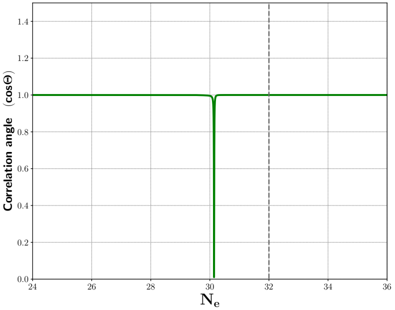
$$\Sigma_{\phi\pi} = 2^{2\left(\nu-\frac{3}{2}\right)}\left[\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2\left(\frac{H}{2\pi}\right)^2\left(\frac{3}{2}-\nu\right) T^{2\left(\frac{3}{2}-\nu\right)}\left[1+\frac{2(5-2\nu)}{4(\nu-1)(3-2\nu)}T^2+\ldots\right]$$

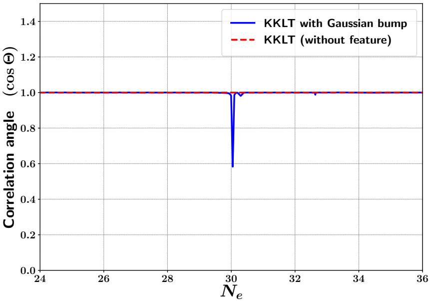
$$\Sigma_{\pi\pi} = 2^{2\left(\nu-\frac{3}{2}\right)}\left[\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2\left(\frac{H}{2\pi}\right)^2\left(\frac{3}{2}-\nu\right)^2 T^{2\left(\frac{3}{2}-\nu\right)}\left[1+\frac{2(7-2\nu)}{4(\nu-1)(3-2\nu)}T^2+\ldots\right]$$

Correlation

$$\gamma = \frac{|\mathrm{Re}(\Sigma_{\phi\pi})|}{\sqrt{\Sigma_{\phi\phi}\Sigma_{\pi\pi}}}$$

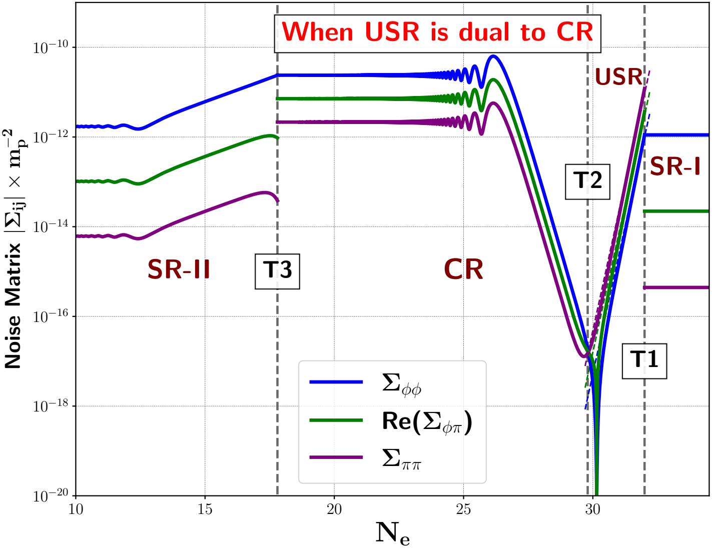
With $\gamma^2 = 1 - \det(\Sigma_{\mathrm{ij}})/(\Sigma_{\phi\phi}\Sigma_{\pi\pi})$



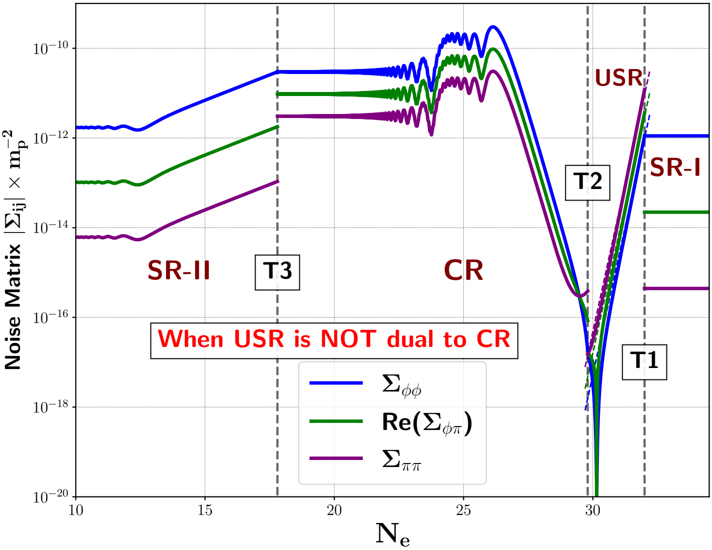


Multiple Transitions

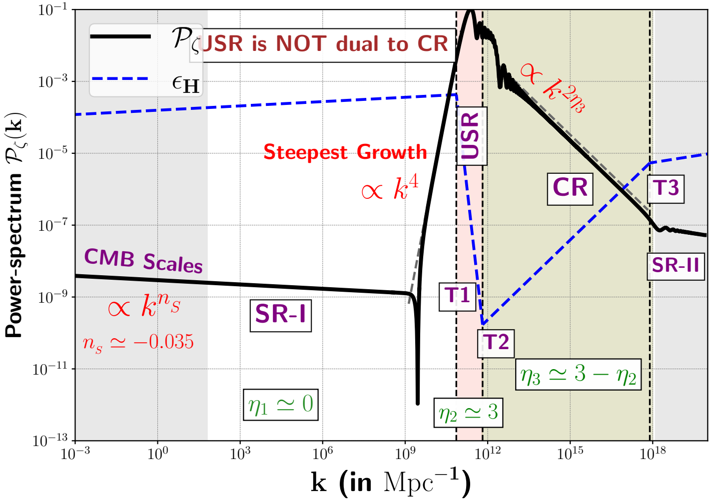
Noise-Matrix: Three Transitions



Noise-Matrix: No Duality



Power-spectrum: No Duality



From $\mathcal{N} \longrightarrow \zeta_{\rm cg}$ (Stochastic $\delta\mathcal{N}$ Formalism)

By **Stochastic $\delta\mathcal{N}$ formalism**, $\boxed{\zeta_{\rm cg} \equiv \zeta(\Phi) = \mathcal{N} - \langle \mathcal{N}(\Phi) \rangle}$

Average no. of e − folds $\langle \mathcal{N}(\Phi) \rangle = \int_0^\infty \mathcal{N}\, P_\Phi(\mathcal{N})\, {\rm d}\mathcal{N}$

$$\Rightarrow \boxed{\langle \mathcal{N}(\Phi) \rangle = \sum_n \frac{\mathcal{A}_n(\Phi)}{\Lambda_n^2}}$$

Threshold $\boxed{\mathcal{N}_c = \zeta_c + \langle \mathcal{N}(\Phi) \rangle = \zeta_c + \sum_n \frac{\mathcal{A}_n(\Phi)}{\Lambda_n^2}}$

Relevance for PBH Mass Function

$$\boxed{\beta(\Phi) \equiv \int_{\zeta_c}^{\infty} P(\zeta_{\rm cg})\,\mathrm{d}\zeta_{\rm cg} = \int_{\mathcal{N}_c}^{\infty} P_{\Phi}(\mathcal{N})\,\mathrm{d}\mathcal{N}}$$

With $\boxed{P_{\Phi}(\mathcal{N}) = \sum_n A_n\, e^{-\Lambda_n \mathcal{N}}}$, $\boxed{\mathcal{N}_c = \zeta_c + \langle \mathcal{N}(\Phi) \rangle}$

And $\Rightarrow \boxed{\langle \mathcal{N}(\Phi) \rangle = \sum_m \frac{\mathcal{A}_m(\Phi)}{\Lambda_m^2}}$

We get $\boxed{\beta(\Phi) = \sum_n \frac{\mathcal{A}_n(\Phi)}{\Lambda_n}\, e^{-\Lambda_n\,[\zeta_c + \langle \mathcal{N}(\Phi) \rangle]}}$

PBH Mass Function: Gaussian vs Non-Gaussian

Stochastic NG

$$\beta^{\mathrm{NG}}(\Phi) = \sum_n \frac{\mathcal{A}_n(\Phi)}{\Lambda_n} e^{-\Lambda_n \left[\zeta_c + \langle \mathcal{N}(\Phi) \rangle\right]}$$

Classical Gaussian

$$\beta^{\mathrm{G}}(\Phi) = \frac{\sigma_{\mathrm{cg}}}{\sqrt{2\pi}\zeta_c} e^{-\frac{\zeta_c^2}{2\sigma_{\mathrm{cg}}^2}}$$

With

$$\sigma_{\mathrm{cg}}^2(\Phi) = \int_{k(\Phi)}^{k_e} \frac{\mathrm{d}k}{k} \mathcal{P}_\zeta(k)$$

Gaussian approx. MIGHT under-estimate PBH abundance by several orders of magnitude