Super-horizon Resonant Magnetogenesis during Inflation

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Outline

- Introduction
- Super-horizon resonant model
- Phenomenological validity
- Conclusions/Future Directions

Introduction

- Magnetic field on the intergalactic scales $\sim \mathrm{Mpc}$

Measured to be ~ $10^{-17} - 10^{-14}$ G

Ando and Kusenko 10, Tavecchio et al. 10
and many others

$$\boxed{B_0 \sim 10^{-15}\ \mathrm{G} \Leftrightarrow \Omega_{\mathrm{B}}^{\mathrm{Obs}} \sim 10^{-23}}$$

- Inflation may be a working regime to explain this $\Omega_{\mathrm{B}}^{\mathrm{Obs}} \sim 10^{-23}$

- There are many magnetogenesis models during inflation

$$\mathscr{L}_{\phi A} = \frac{1}{4} f(\phi)^2 F_{\mu\nu} F^{\mu\nu} \qquad \mathscr{L}_{\phi A} = \frac{1}{4} I(\phi) F_{\mu\nu} \tilde{F}^{\mu\nu} \qquad + \quad \cdots$$

Conformal invariance is broken

Maxwell: $\Omega_{\mathrm{B}} \propto (k/a)^4$ $\Rightarrow$ Exponentially decreases by many e-folding numbers during inflation

Introduction: Obstacles

Some obstacles may arise during the production from inflation

- Strong-coupling problem

$$\mathscr{L} = \frac{1}{4} f(\phi)^2 F_{\mu\nu} F^{\mu\nu} + e\bar{\psi}\gamma^\mu\psi A_\mu + i\bar{\psi}\gamma^\mu\partial_\mu\psi \xRightarrow[A^c_\mu = fA_\mu]{\text{Canonically normalized}} \frac{e}{\underline{f}}\bar{\psi}\gamma^\mu\psi A^c_\mu$$

If $f^2 \ll 1$ during the production, the perturbative calculation breaks down

- Back-reaction problem

The gauge field A_μ is enhanced a lot $\Rightarrow$ $\Omega_\mathrm{B}, \Omega_\mathrm{E}, \Omega_\mathrm{others} \gg 1$: Destroy inflation

$$\rho_\mathrm{tot} = \rho_\phi = 3H^2 M_\mathrm{Pl}^2$$

Gasperini et al. 95, Demozzi et al. 09, Fujita and Mukohyama 12, Green and Kobayashi 16, Moghaddam et al. 17 and many others

Introduction

- Resonant production in $f(\phi)F\tilde{F}$ model $\mathscr{L}_{\phi A} = \frac{1}{4} f(\phi) F_{\mu\nu} \tilde{F}^{\mu\nu}$ Byrnes et al. 12

Equation of motion: $' = d/d\eta$ $h = \pm 1$

$$A_h''(\eta, \boldsymbol{k}) + \left[k^2 + \underline{hkf'}\right] A_h(\eta, \boldsymbol{k}) = 0$$

On large scales ($k \to 0$): no coupling of ϕ and A

During inflation $\phi = \phi(t)$: $f(\phi) \to f(t)$ $a \sim e^{Ht} \sim -1/(H\eta)$

Assume: $f(\eta) = \gamma \sin(2\omega\eta)$ $z = \omega\eta$ $q = \gamma k/\omega = \gamma\sqrt{p}$ $A \propto e^{\mu_k z}$

$$\Rightarrow \boxed{\frac{d^2A(z)}{dz^2} + \left[p - 2q\cos(2z)\right]A(z) = 0}$$

Mathieu equation

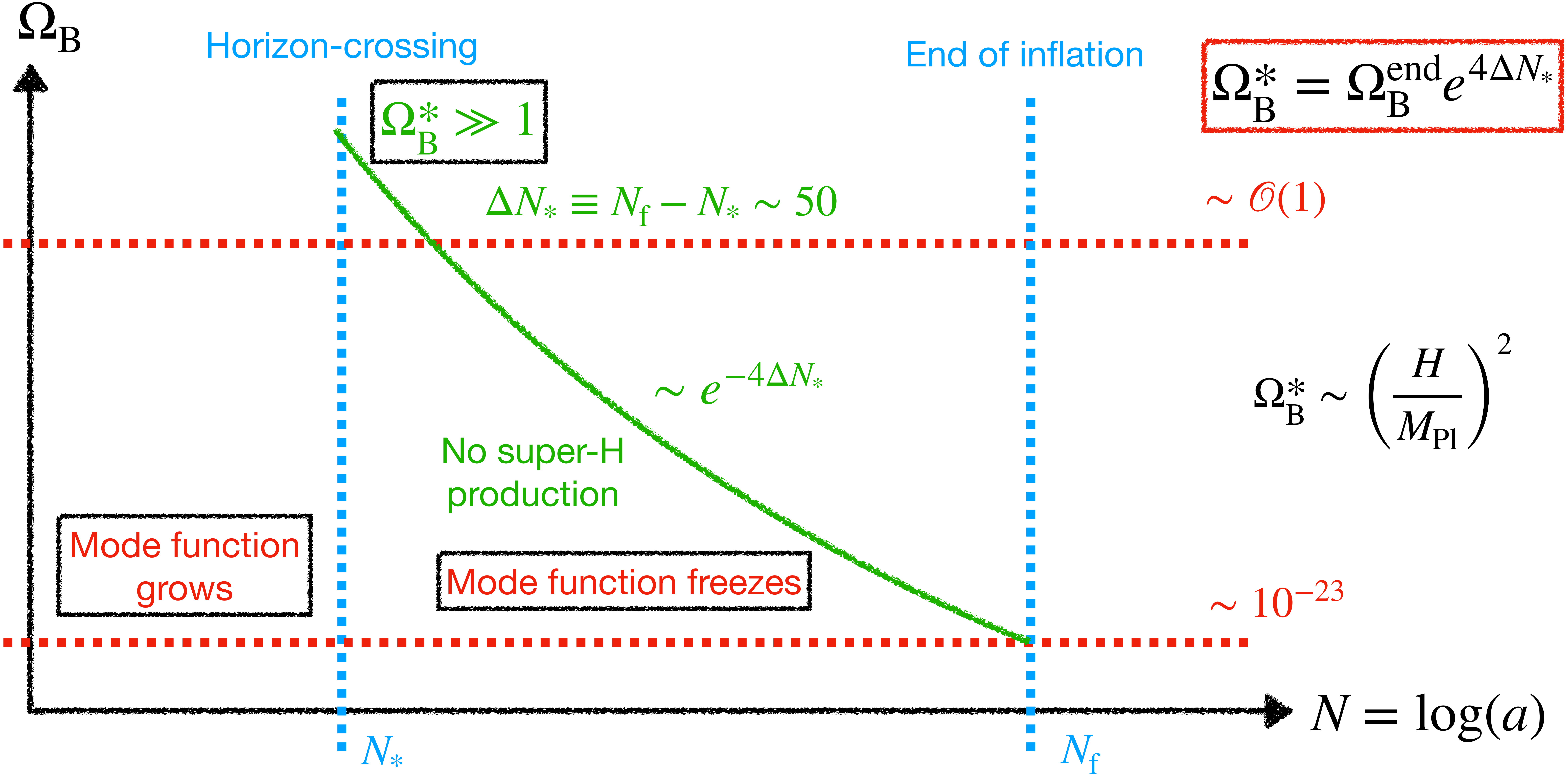
Sup-horizon amplification

- Resonant model from axion-like coupling $\phi F\tilde{F}$ during inflation and preheating (Patel, Tashiro and Urakawa 19)

Introduction: No-go theorem

$$\boxed{B_0 \sim 10^{-15}\ \mathrm{G} \Leftrightarrow \Omega_\mathrm{B} \sim 10^{-23}}$$

$$\boxed{\Omega_\mathrm{B}^* = \Omega_\mathrm{B}^\mathrm{end} e^{4\Delta N_*}}$$



Horizon-crossing

End of inflation

$\boxed{\Omega_\mathrm{B}^* \gg 1}$

$\Delta N_* \equiv N_\mathrm{f} - N_* \sim 50$

$\sim \mathcal{O}(1)$

$\sim e^{-4\Delta N_*}$

$$\Omega_\mathrm{B}^* \sim \left(\frac{H}{M_\mathrm{Pl}}\right)^2$$

No super-H production

Mode function grows

Mode function freezes

$\sim 10^{-23}$

Super-horizon resonant model

- The action $S = \int \mathrm{d}^4x\sqrt{-g}\left[\mathscr{L}_{\rm EH} + \mathscr{L}_\phi + \mathscr{L}_{\phi A}\right]$

$$e_{\rm eff}^2 = f(\phi)^{-2}$$

$$\mathscr{L}_{\rm EH} = \frac{M_{\rm Pl}^2}{2}R \qquad \underline{\mathscr{L}_\phi} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \qquad \mathscr{L}_{\phi A} = \underline{f(\phi)^2}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m_A^2 A_\mu A^\mu\right]$$

Inflaton

Breaks conformal invariance

- Background: $\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\mathrm{d}\boldsymbol{x}^2$ and $\phi = \phi(t)$

$$3H^2M_{\rm Pl}^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \cancel{\rho}_{\rm em} \qquad \ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = I(\cancel{A}_\mu)$$

Assume slow-roll potential $a(t) \sim \exp(Ht) \simeq$ de-Sitter

Super-horizon resonant model

- The production of A_μ during inflation driven by $\phi(t)$
- Introduce the long. mode via $A_\mu \to \underline{A_\mu} + \partial_\mu \chi$

$$A_0(t,\boldsymbol{x}) = 0 \quad \text{and} \quad \delta^{ij}\partial_i A_j(t,\boldsymbol{x}) = 0$$

The coupling function:
$\Rightarrow f = f(t)$

Fourier mode:
$$A_i(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{\lambda=\pm} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}_i^\lambda(\hat{k})\left[A_\lambda(t,k)\hat{b}^\lambda_{\boldsymbol{k}} + A_\lambda(t,k)^*\hat{b}^{\lambda\dagger}_{-\boldsymbol{k}}\right]$$

EOM:
$\mathscr{A} \equiv \sqrt{F}A$

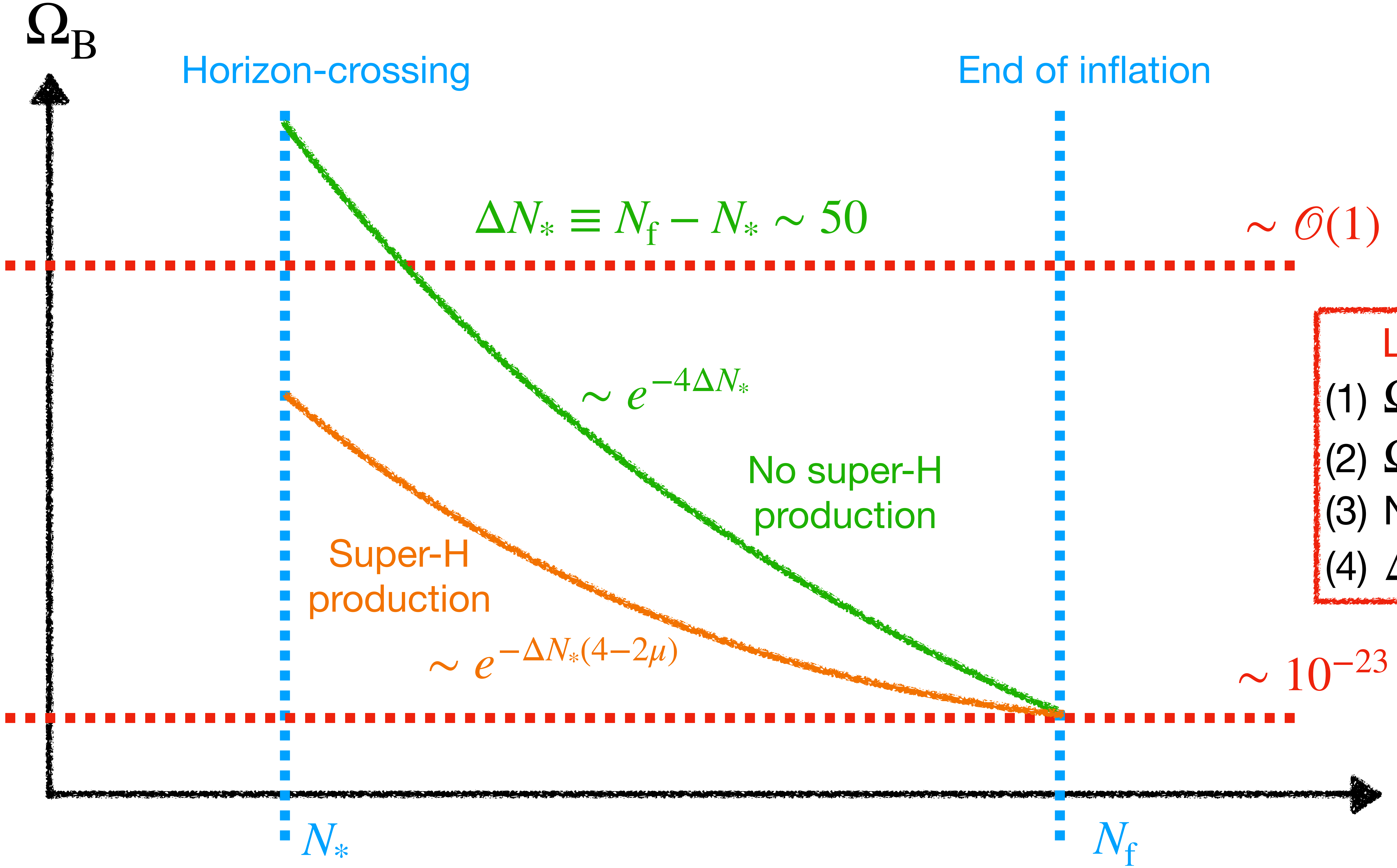
$$\boxed{\ddot{\mathscr{A}} + \left[\frac{k^2}{a^2} + m_A^2 + \frac{1}{4}\left(\frac{\dot{F}}{F}\right)^2 - \frac{\ddot{F}}{2F}\right]\mathscr{A} = 0}$$

$$F \equiv a(t)f^2$$

Resonant model: Main idea

$B_0 \sim 10^{-15}\,\mathrm{G} \Leftrightarrow \Omega_\mathrm{B} \sim 10^{-23}$

$\Omega_\mathrm{B} \equiv \rho_\mathrm{B}/\rho_\mathrm{tot}$



List of constraints:

(1) $\Omega_\mathrm{B}(N_\mathrm{f}) \sim 10^{-23}$

(2) $\Omega_\mathrm{B}, \Omega_\mathrm{M}, \Omega_\mathrm{E} < 1$

(3) No strong coupling

(4) ΔN_CMB large enough

Super-horizon resonant model

- Our model: F oscillates with frequency ω during inflation

$$\frac{F'}{F} = -2\gamma\sin(\underline{2z})$$

Positive definite

$${}' = \frac{d}{dz}$$

$z = \omega(t - t_{\rm i})/2$

$t_{\rm i}$: onset of oscillations

- EOM of $\mathscr{A}$ becomes $a(z) = e^{2rz}$ $r = H/\omega$

Whittaker-Hill equation

$$\boxed{\mathscr{A}'' + \Big[C(z) + 2q\cos(4z) + 2p\cos(2z)\Big]\mathscr{A} = 0}$$

$$C(z) \equiv \frac{4k^2}{a^2\omega^2} - \frac{\gamma^2}{2} + \underline{\delta} \qquad \delta = \frac{4m_A^2 r^2}{H^2} \qquad p \equiv \gamma \qquad q \equiv \frac{1}{4}\gamma^2$$

Super-horizon resonant model

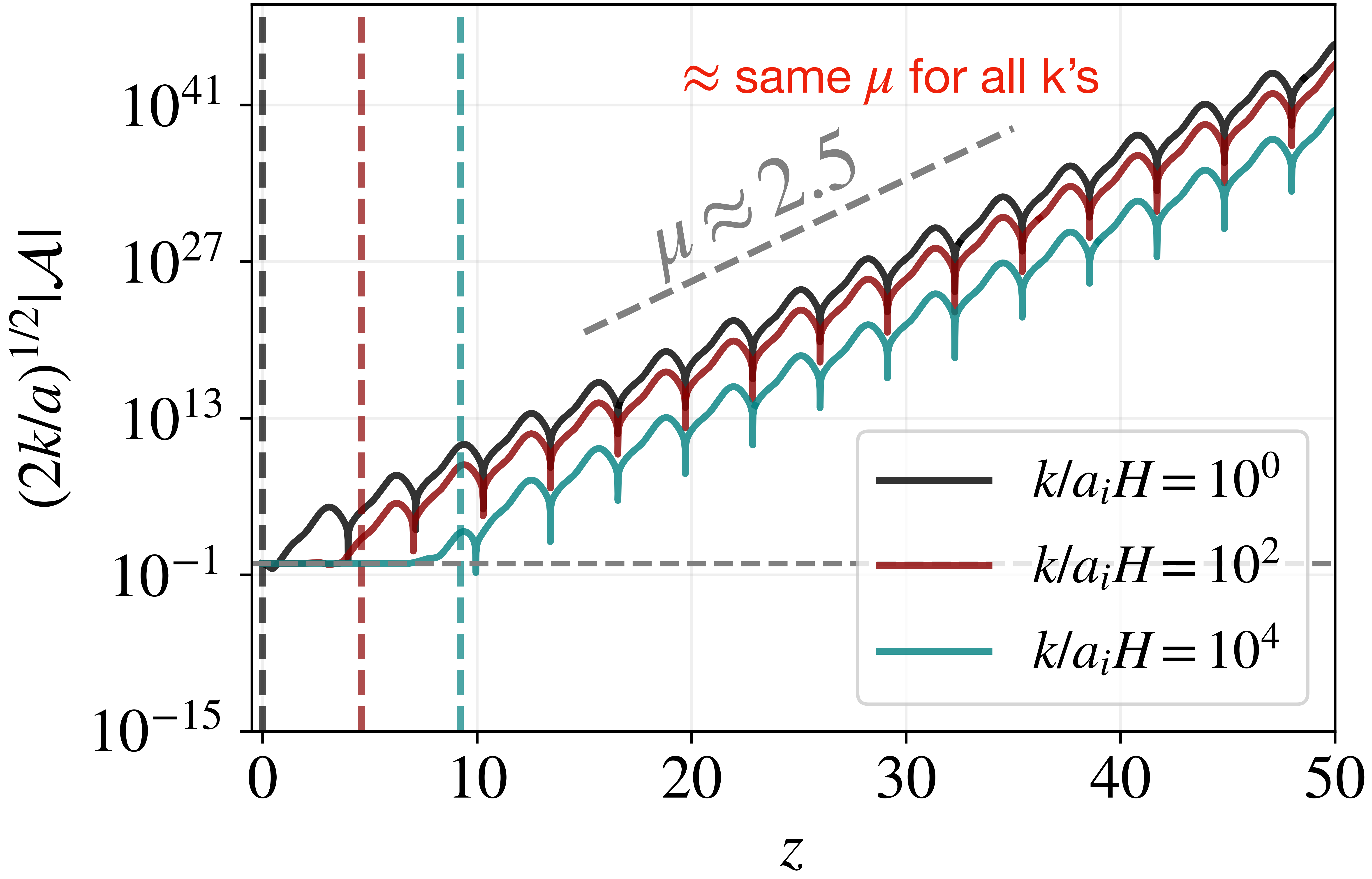
- Numerical solutions with different k

$z_{\rm i} = 0$ and $z_{\rm f} = 50$

$$z_{\rm f} - z_{\rm i} = \frac{N_{\rm f} - N_{\rm i}}{2r}$$

$\delta = 10^{-2}$, $r = 0.5$, and $\gamma = 7$

≈ 5.3



The mode crosses the horizon at

$$z_c = \frac{1}{2r}\log\left(\frac{k}{\omega r}\right)$$

The expo. growth happens when the mode exits the horizon

Super-horizon amplification

Phenomenological validity

Corresponding k at the end of inflation

- Estimate the magnetic field spectrum

We use $|\sqrt{2k}\mathscr{A}| \sim \exp(\mu \underline{\Delta z_k})$ The time the mode spends outside the horizon

$$\Delta z_k = \frac{1}{2r} \log\left(\frac{k_\mathrm{f}}{k}\right)$$

⇒ The mode that exits horizon earlier will experience amplification longer

⇒ No amplification for $k = k_\mathrm{f}$

$$\frac{\mathrm{d}\Omega_\mathrm{B}}{\mathrm{d}\log k} = \mathscr{C}\left(\frac{H}{M_\mathrm{Pl}}\right)^2 \left(\frac{k}{k_\mathrm{f}}\right)^{5-\mu/r} \left\{ \begin{array}{l} 5-\mu/r = 0 : \text{Scale-invariant } \Omega_\mathrm{B} \\ 5-\mu/r > 0 : \text{Blue-tilted } \Omega_\mathrm{B} \\ 5-\mu/r < 0 : \text{Red-tilted } \Omega_\mathrm{B} \end{array} \right.$$

- For $\mu/r = 1 \Rightarrow \mathrm{d}\Omega_\mathrm{B}/\mathrm{d}\log k \sim k^4$ (Maxwell theory)

Phenomenological validity

- Energy density in the mass term

We use $|\sqrt{2k}\mathscr{A}| \sim \exp(\mu \Delta z_k)$ $\Rightarrow$ $\dfrac{d\Omega_{\mathrm{M}}}{d\log k} \simeq \dfrac{\delta}{r^2}\left(\dfrac{H}{M_{\mathrm{Pl}}}\right)^2\left(\dfrac{k}{k_{\mathrm{f}}}\right)^{3-\mu/r}$

- Require that (1) $\Omega_{\mathrm{B}}(t_{\mathrm{f}}) \gtrsim 10^{-23}$ $n_{\mathrm{B}} \equiv 5 - \mu/r$

$$2x + n_{\mathrm{B}}(\kappa - \Delta N_{\mathrm{CMB}}) \gtrsim -23\log(10)$$

The CMB mode at the horizon crossing

$x \equiv \log(H/M_{\mathrm{Pl}})$ $x_0 \equiv \log(H_0/M_{\mathrm{Pl}})$ $\kappa \equiv \log(k_{\mathrm{B}}/k_{\mathrm{CMB}})$ $\Delta N_{\mathrm{CMB}} \equiv \log(a_{\mathrm{f}}/a_{\mathrm{CMB}})$

$x_0 = -140$ for
$H_0 \sim 10^{-33}$ eV

$\kappa \sim 7$ for
$k_{\mathrm{B}} \sim \mathrm{Mpc}^{-1} \sim 10^3 k_{\mathrm{CMB}}$

Phenomenological validity

(2) No $\Omega_{\rm B}$ back-reaction: $\Omega_{\rm B} < 1 \quad \Longrightarrow \quad 2x + n_{\rm B}(\kappa - \Delta N_{\rm CMB}) \lesssim 0$

(3) No $\Omega_{\rm M}$ back-reaction: $\Omega_{\rm M} < 1 \quad \Longrightarrow \quad 2x + (n_{\rm B} - 2)(\kappa - \Delta N_{\rm CMB}) \lesssim 0$

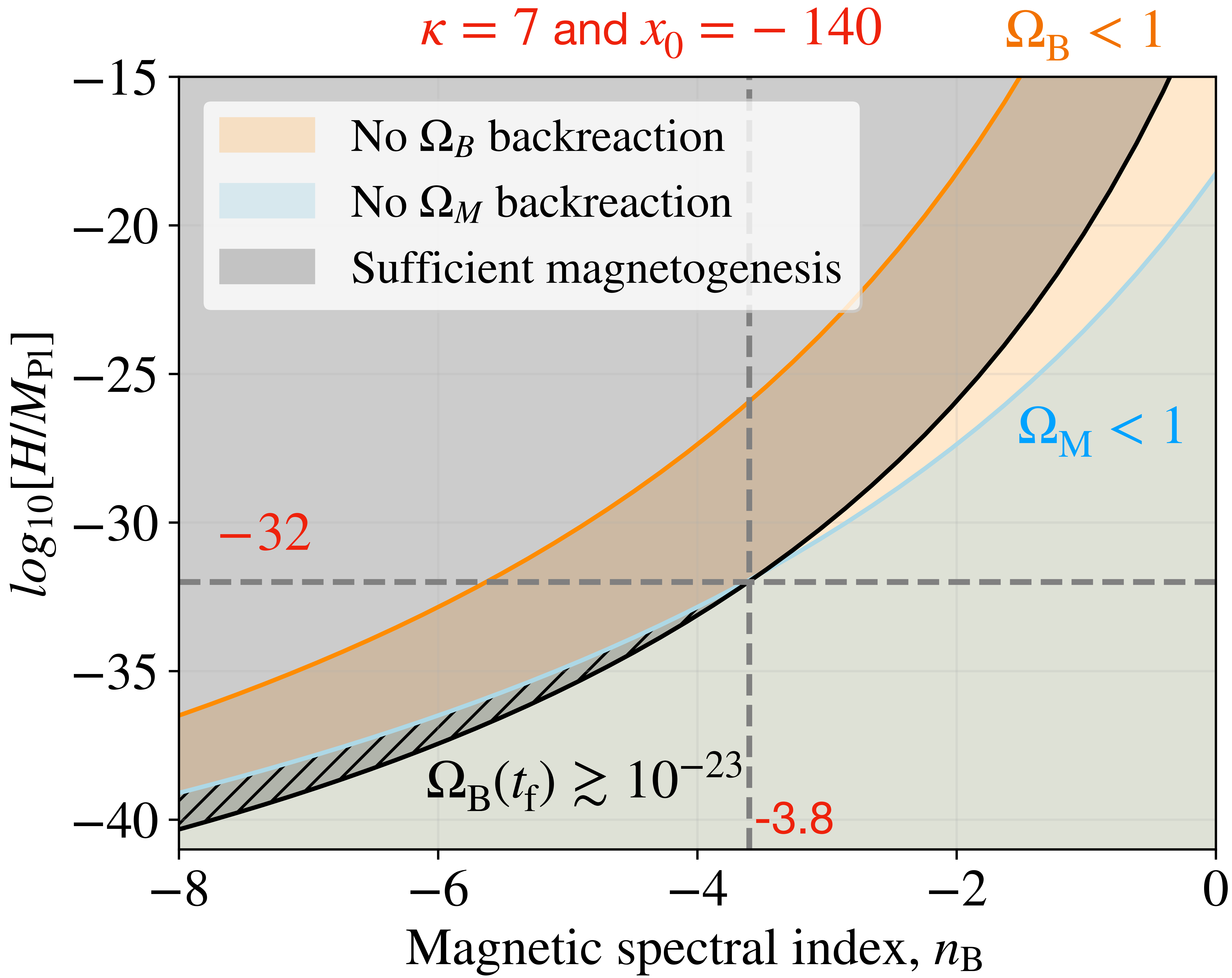
(4) Successful inflation requires at least $\Delta N_{\rm CMB} = \dfrac{1}{2}(x - x_0)$

$(a_0 H_0)^{-1} < (a_{\rm CMB} H)^{-1}$

All three constraints become

$$\left.\begin{array}{l} (1)\ \Omega_{\rm B}(t_{\rm f}) \gtrsim 10^{-23} \\ (2)\ \Omega_{\rm B} < 1 \\ (3)\ \Omega_{\rm M} < 1 \end{array}\right\} \quad \begin{array}{l} (4 - n_{\rm B})x + n_{\rm B}(2\kappa + x_0) \gtrsim -46\log(10) \\ (4 - n_{\rm B})x \lesssim -n_{\rm B}(2\kappa + x_0) \\ (6 - n_{\rm B})x \lesssim -(n_{\rm B} - 2)(2\kappa + x_0) \end{array}$$

Phenomenological validity


$\kappa = 7$ and $x_0 = -140$

$$\boxed{n_B \equiv 5 - \mu/r}$$

Allowed region: $n_B \lesssim -4$ and

$H/M_{Pl} \lesssim 10^{-32}$

Low-scale inflation

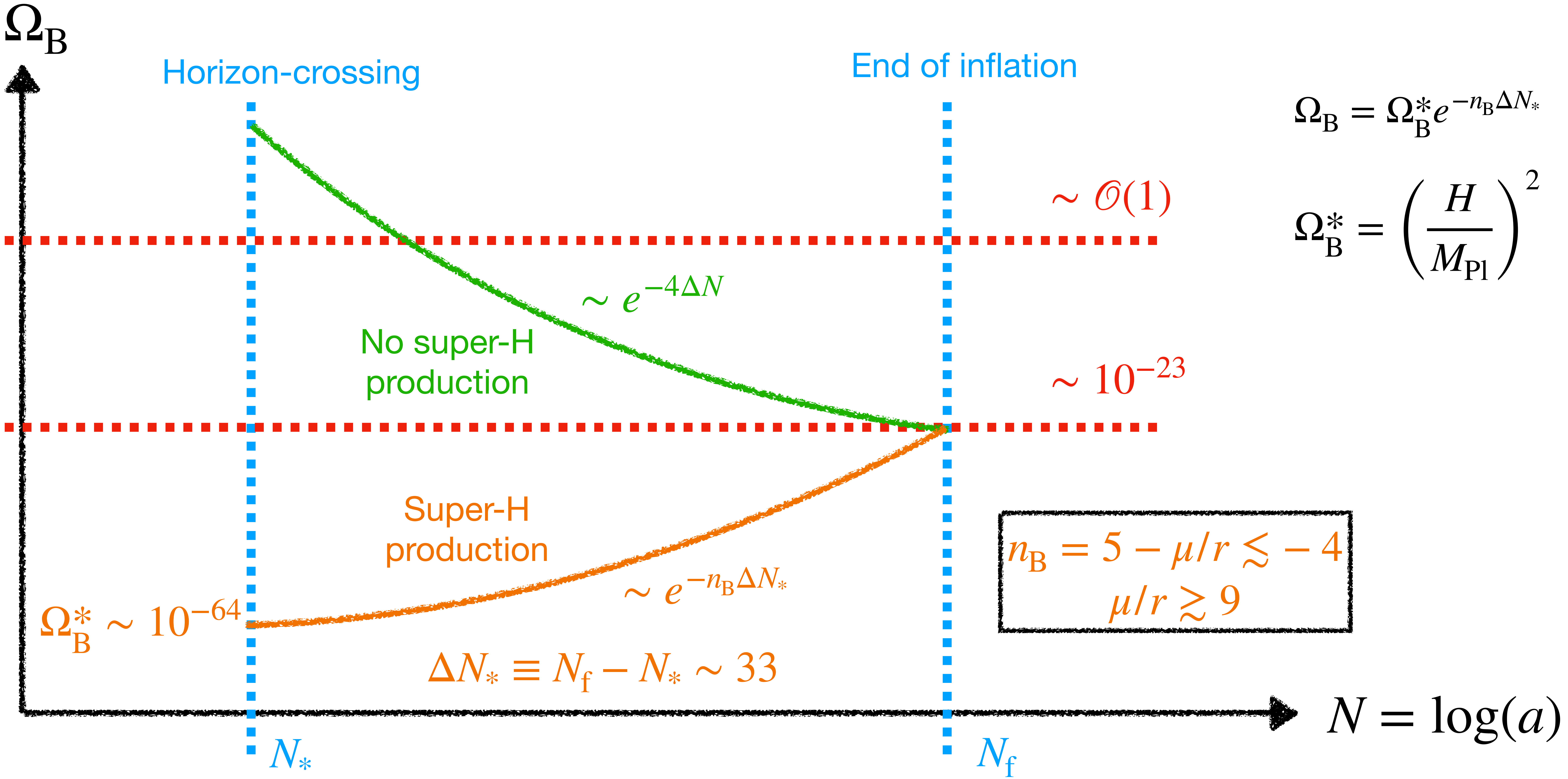
$$H/M_{Pl} \sim 10^{-32} \Rightarrow T_R \sim 10^3 \text{ GeV}$$

$$\Delta N_{CMB} \sim 33$$

Both scale-invariant ($n_B = 0$) and blue-tilted ($n_B > 0$) cases are excluded

Resonant model: Summary

$\boxed{B_0 \sim 10^{-15}\ \mathrm{G} \Leftrightarrow \Omega_\mathrm{B} \sim 10^{-23}}$



$\Delta N_* \equiv N_\mathrm{f} - N_* \sim 33$

$$\Omega_\mathrm{B} = \Omega_\mathrm{B}^* e^{-n_\mathrm{B}\Delta N_*}$$

$$\Omega_\mathrm{B}^* = \left(\frac{H}{M_\mathrm{Pl}}\right)^2$$

$$\boxed{\begin{gathered} n_\mathrm{B} = 5 - \mu/r \lesssim -4 \\ \mu/r \gtrsim 9 \end{gathered}}$$

Phenomenological validity

- No strong coupling problem $F \equiv a(t)f^2$ $e_{\text{eff}}^2 = f(\phi)^{-2}$ $\Rightarrow$ $e_{\text{eff}}^2 \sim \frac{a}{F}$

  $\Rightarrow$ We can start from $e_{\text{eff}}^2 \ll 1$at t_{i} and dynamically evolve it toward $e_{\text{eff}}^2 \sim \mathcal{O}(1)$ at t_{f}

  (f can be rescaled arbitrarily)

  Standard parametrization: $f \sim a^{\alpha}$. For $\alpha = 2$ (scale invariance), $e_{\text{eff}}^2 \sim a^{-4}$

  Having weak coupling regime during inflation $\Rightarrow$ $e_{\text{eff}}^2 \sim \mathcal{O}(1)$ at t_{i} and evolve to $e_{\text{eff}}^2 \ll 1$ at t_{f}

- No Ω_{E} back-reaction

  Ω_{E} and Ω_{M} scale in the same way $\Omega_{\text{M}} \sim \Omega_{\text{E}} \sim |\text{Mode function}|^2/a^2$

  $\Rightarrow$ Keeping $\Omega_{\text{M}} < 1$ during the production $\Rightarrow$ no Ω_{E} back-reaction

  The suppression of Ω_{E} during inflation may be due to the Schwinger pair production
  (Dunne and Hall 98, Frob et al. 14)

Conclusions/Future Directions

Conclusions

- We constructed a resonant model for magnetic field production
- The amplification happens on super-horizon scales
- We obtained the allowed region in our parameter space compatible with all constraints

Future directions

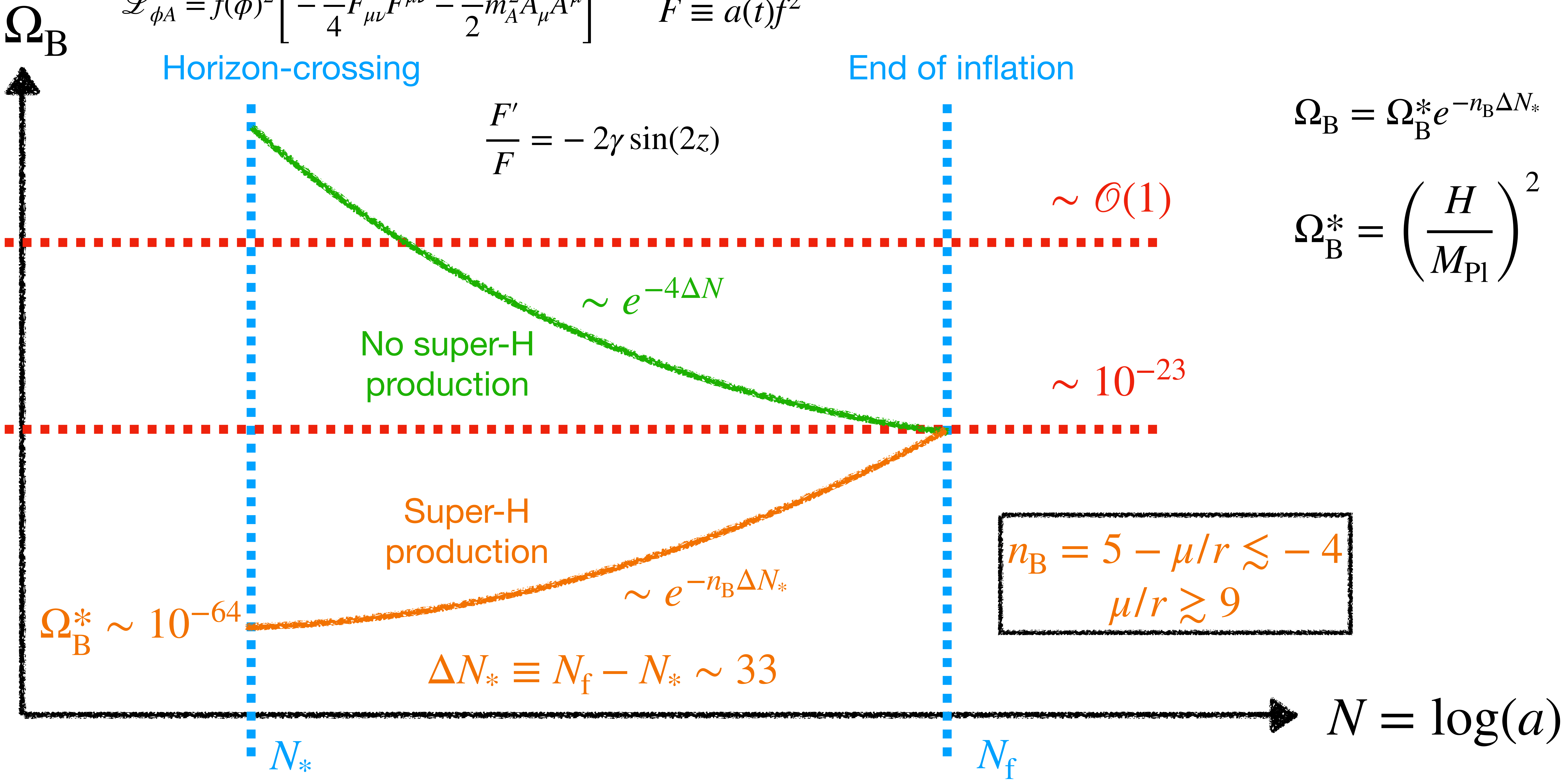
- Study the production of primordial GW in this resonant model
- Study a similar mechanism during reheating

Backup

Resonant model: Summary

$\boxed{B_0 \sim 10^{-15}\ \mathrm{G} \Leftrightarrow \Omega_\mathrm{B} \sim 10^{-23}}$

$$\mathscr{L}_{\phi A} = f(\phi)^2\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m_A^2 A_\mu A^\mu\right] \qquad F \equiv a(t)f^2$$



Ω_B

Horizon-crossing

End of inflation

$$\frac{F'}{F} = -2\gamma\sin(2z)$$

$\sim \mathcal{O}(1)$

$\sim e^{-4\Delta N}$

No super-H production

$\sim 10^{-23}$

Super-H production

$\sim e^{-n_\mathrm{B}\Delta N_*}$

$\Omega_\mathrm{B}^* \sim 10^{-64}$

$\Delta N_* \equiv N_\mathrm{f} - N_* \sim 33$

$N = \log(a)$

N_*

N_f

$$\Omega_\mathrm{B} = \Omega_\mathrm{B}^* e^{-n_\mathrm{B}\Delta N_*}$$

$$\Omega_\mathrm{B}^* = \left(\frac{H}{M_\mathrm{Pl}}\right)^2$$

$$\boxed{\begin{gathered} n_\mathrm{B} = 5 - \mu/r \lesssim -4 \\ \mu/r \gtrsim 9 \end{gathered}}$$

Super-horizon resonant model

- Comment on the massless mode $m_A^2 = 0$

$$\boxed{\ddot{A} + \frac{\dot{F}}{F}\dot{A} + \frac{k^2}{a^2}A + \cancel{m_A^2}A = 0} \xRightarrow{k \to 0} (F\dot{A})^{\cdot} = 0 \Rightarrow A \sim const. + \int \frac{const.}{F} dt$$

No exponentially amplification of A if F is a bounded function

- Introduce $\mathscr{A} \equiv \sqrt{F}A$

$$\ddot{A} + \frac{\dot{F}}{F}\dot{A} + \frac{k^2}{a^2}A + m_A^2 A = 0 \Rightarrow \boxed{\ddot{\mathscr{A}} + \left[\frac{k^2}{a^2} + m_A^2 + \frac{1}{4}\left(\frac{\dot{F}}{F}\right)^2 - \frac{\ddot{F}}{2F}\right]\mathscr{A} = 0}$$

For $f(\phi)^2 F\tilde{F}$: the coupling of $\mathscr{A}\phi \propto k \Rightarrow$ suppressed enhancement on large scales

Super-horizon resonant model

Energy densities: $\rho_{\mathrm{em}} = \langle T^{(A)}_{tt} \rangle = \frac{f^2}{2} \langle E_\mu E^\mu + B_\mu B^\mu + m_A^2 A_\mu A^\mu \rangle \equiv \rho_{\mathrm{E}} + \rho_{\mathrm{B}} + \rho_{\mathrm{M}}$

$E_\mu \equiv F_{\mu\nu} u^\nu \qquad B_\mu \equiv \tilde{F}_{\mu\nu} u^\nu \qquad u^\mu = (1, \mathbf{0})$

$$\frac{\mathrm{d}\rho_{\mathrm{B}}}{\mathrm{d}\log k} = \frac{1}{4\pi^2} \left(\frac{k}{a}\right)^4 \frac{1}{a} |\sqrt{2k}\mathscr{A}(t,k)|^2 \xRightarrow{\text{Conformal inv. case}} \frac{\mathrm{d}\rho_{\mathrm{B}}}{\mathrm{d}\log k} \simeq \frac{1}{4\pi^2} \left(\frac{k}{a}\right)^4$$

Exponentially decreases during inflation

$$\left.\begin{aligned} \frac{\mathrm{d}\rho_{\mathrm{E}}}{\mathrm{d}\log k} &= \frac{f^2}{4\pi^2} \left(\frac{k}{a}\right)^2 \left| \frac{d}{dt} \left(\frac{\sqrt{2k}\mathscr{A}(t,k)}{\sqrt{af}} \right) \right|^2 \\ \frac{\mathrm{d}\rho_{\mathrm{M}}}{\mathrm{d}\log k} &= \frac{m_A^2}{4\pi^2} \left(\frac{k}{a}\right)^2 \frac{1}{a} |\sqrt{2k}\mathscr{A}(t,k)|^2 \end{aligned}\right\}$$

Decrease less compared to ρ_{B} $\Rightarrow$ Strong constraints on our parameters

Introduction: No-go theorem

- The production of $A(\eta, k)$ is peaked at the horizon k_*

$$|\sqrt{2k}A|^2 = 2k_*\bar{A}_*^2\,\delta(k/k_* - 1)$$

After horizon-crossing: the mode $|\sqrt{2k}A|$ freezes and $\Omega_{\rm B} \propto a^{-4}$

$$\boxed{\Omega_{\rm B}^* = \Omega_{\rm B}^{\rm end} e^{4\Delta N_*}} \qquad \Delta N_* = N_{\rm f} - N_*$$

$$\Omega_{\rm B} \equiv \rho_{\rm B}/\rho_{\rm tot}$$

$$\rho_{\rm tot} = \rho_\phi = 3H^2M_{\rm Pl}^2$$

Assume: Rad is right after inflation until today $\Rightarrow$ $\Omega_{\rm B}$ stays const. after inflation

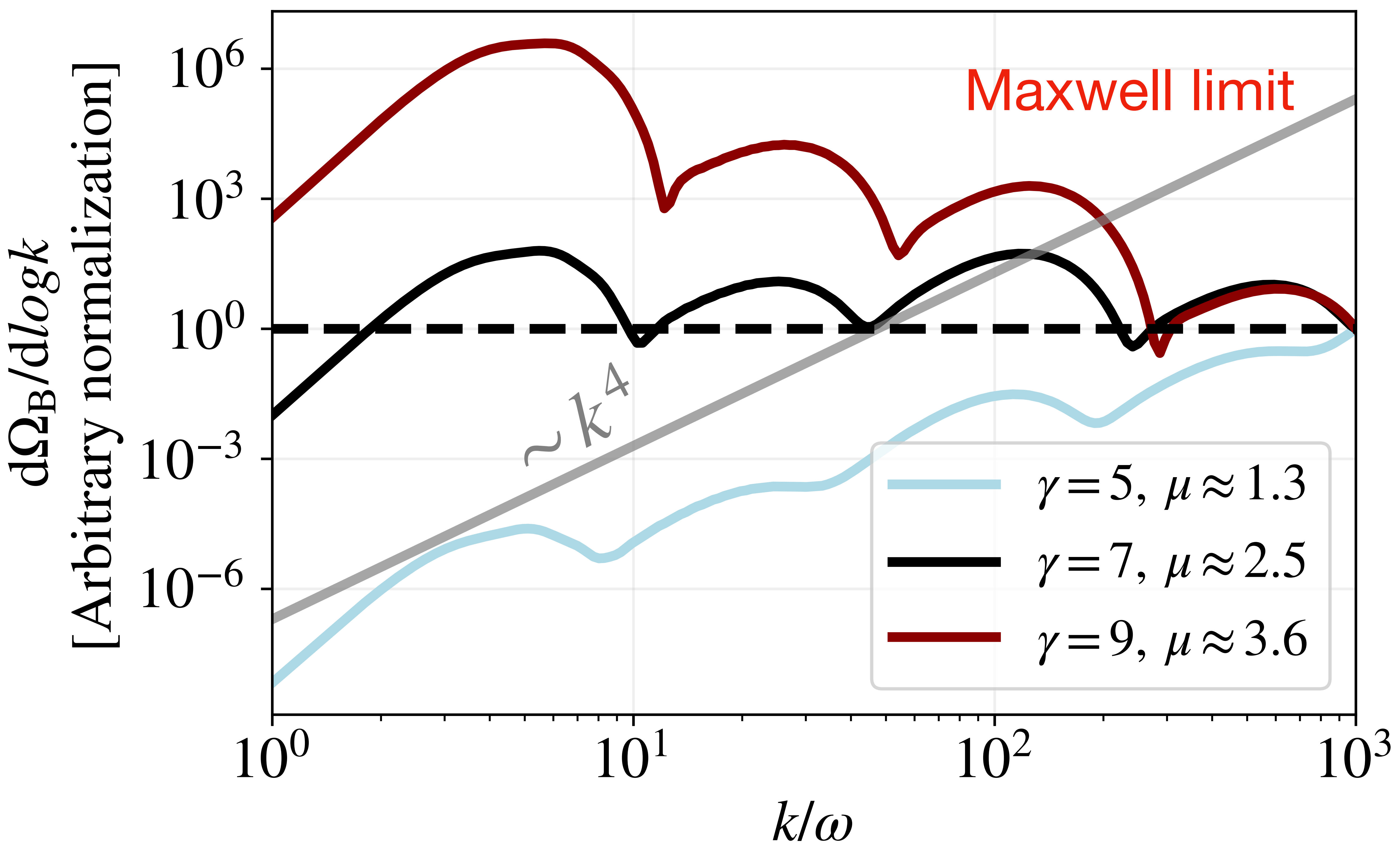
- Requiring $\Omega_{\rm B}^{\rm end} \gtrsim 10^{-23}$ and $\Delta N_* \sim 50$ $\Rightarrow$ $\Omega_{\rm B}^* \gg 1$ **NO-GO**

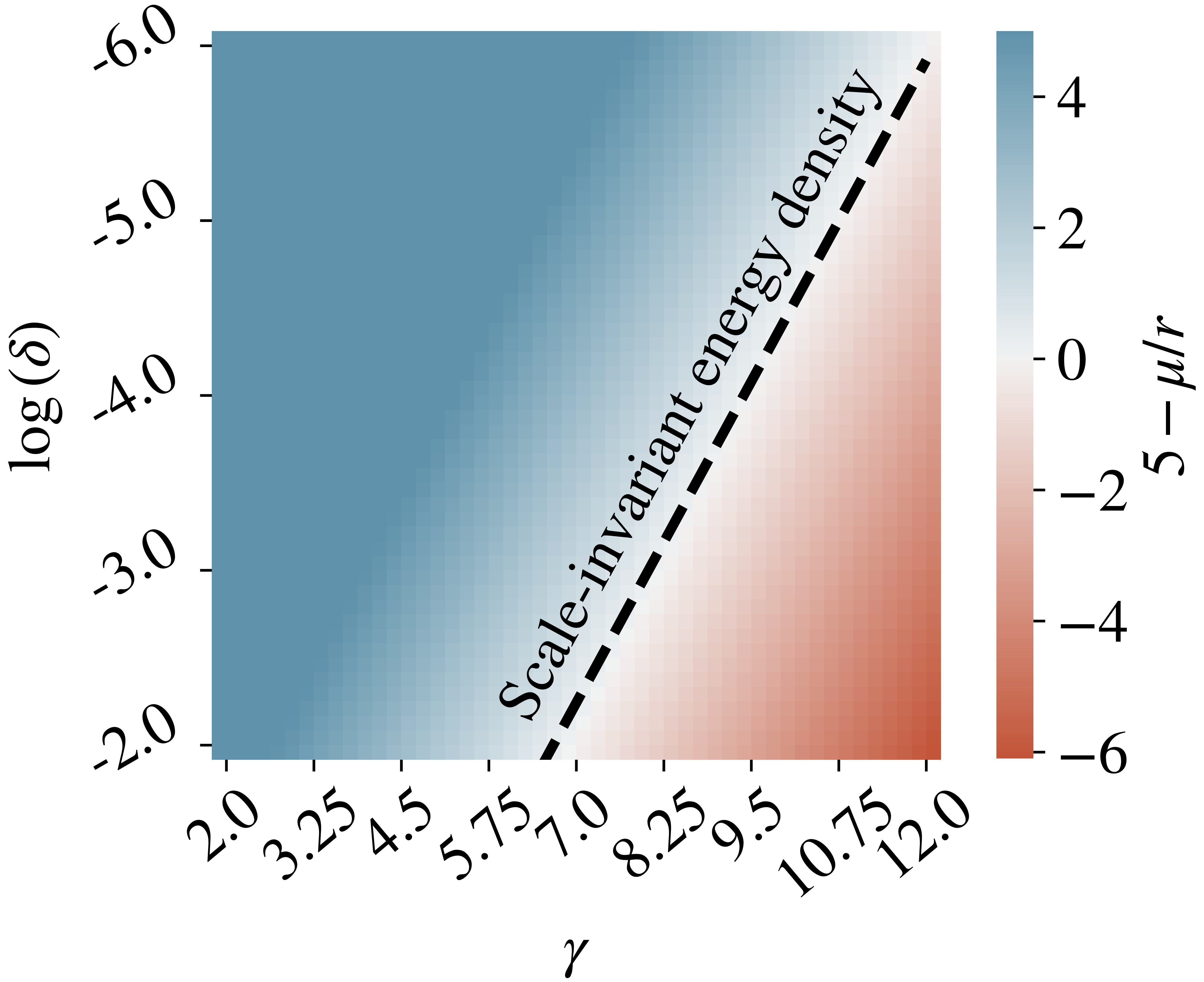
If $\Delta N_* \sim 11$, the back-reaction can be avoided

Durrer et al. 11, Byrnes et al. 12

Phenomenological validity

- Spectra of magnetic field Ω_{B} $\delta = 10^{-2}$, $r = 0.5$, and $\Delta N = 50$





Super-horizon resonant model

- Analytical approximation: neglect z-dependence in $C(z)$

Floquet theorem:

Floquet exponent

$$|\sqrt{2k}\ \mathscr{A}(z)| = e^{\pm\mu z}h(z) \qquad \text{with} \qquad \mu \in \mathbb{C}$$

Periodic function

$\mathrm{Re}(\mu) = 0$: Stable solution

$\mathrm{Re}(\mu) \neq 0$: Unstable solution

The real part is determined by

$$\mu_R(\gamma,\delta) = \frac{1}{\pi}\log\left[D + \sqrt{D^2 - 1}\right] \qquad \text{with} \qquad D(\gamma,\delta) \equiv 2\underline{\Delta(0)}\sinh^2\left(\frac{\pi}{2}\sqrt{|C|}\right) + 1$$

Determinant of infinite matrix

This analytic solution is valid when $k \to 0$

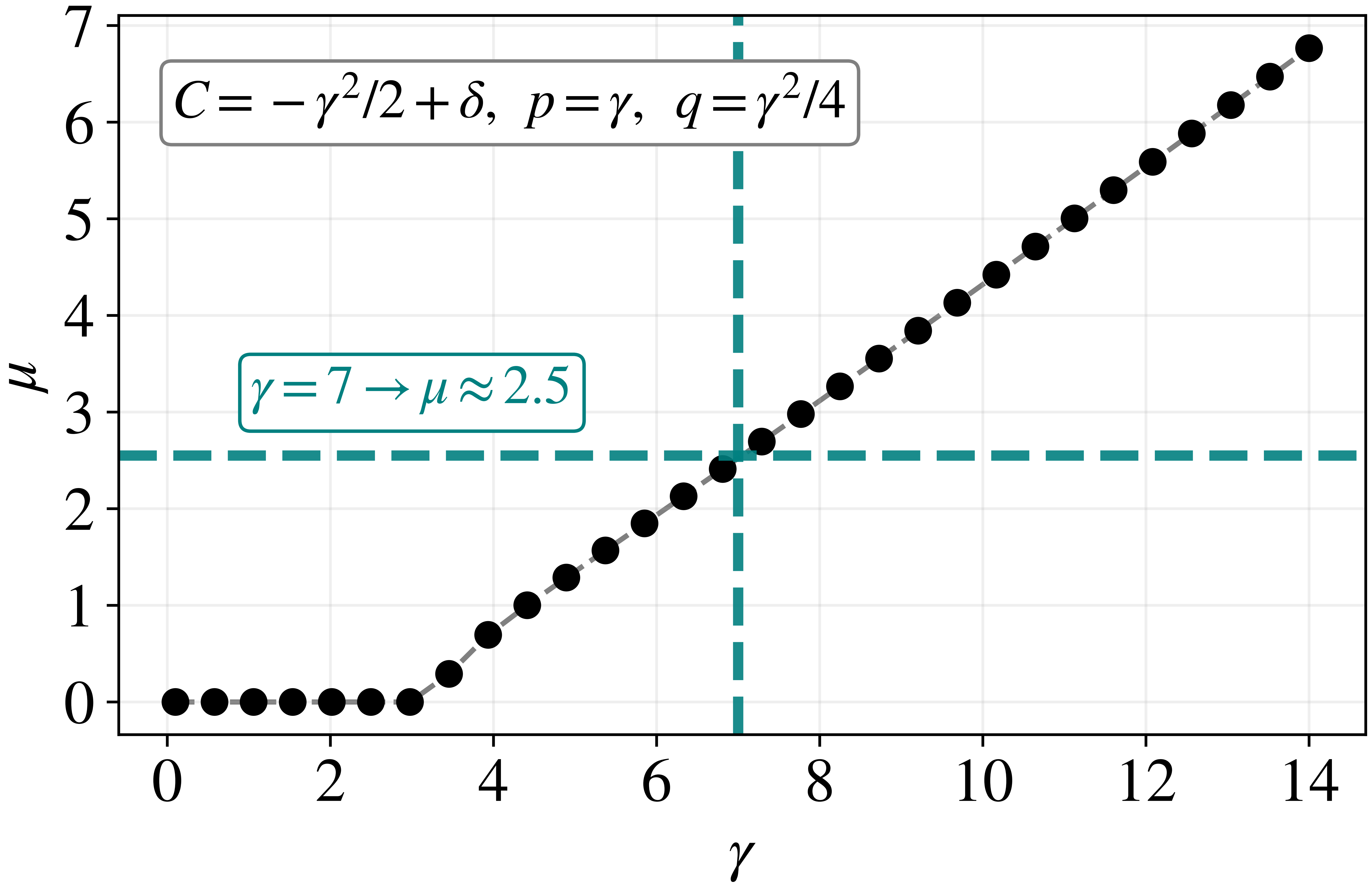
Super-horizon resonant model

- Plot of μ versus γ

$\delta = 10^{-2}$

Dotted: Floquet formula

Dashed: Numerical solutions



$\mu \simeq 2.5$: Scale-invariant Ω_B

$\mu \gtrsim 2.5$: Red tilted Ω_B

$\mu \lesssim 2.5$: Blue-tilted Ω_B

$$\boxed{\mu(\gamma) = max\left\{0, 0.6\gamma - 1.64\right\}}$$

Longitudinal mode and initial conditions

$$\frac{d}{dt}(a^3 f^2 \dot{\chi}_{\rm c}) + Fk^2 \chi_{\rm c} = 0$$

In the limit $k \to 0$ the mode χ is exponentially decreasing

The mode χ does not enter into the definition of magnetic field

We completely neglect the contributions of χ in the energy density

- Initial conditions for $\mathscr{A}(z, k)$

$$|\sqrt{2k}\ \mathscr{A}(z)|_{z=z_{\rm i}} = 1\ , \quad \left|\sqrt{2k}\ \frac{{\rm d}\mathscr{A}(z)}{{\rm d}z}\right|_{z=z_{\rm i}} = \frac{2k}{\omega}$$

Finite reheating

Assume that reheating happens with $H \sim a^{-\beta}$ $\Omega_{\mathrm{B}} \sim a^{2(\beta-2)}$

Note $\beta = 3$ Kination: $w = 1$ and $\rho \sim a^{-6}$

The constraints are modified to

$$(4 - n_{\mathrm{B}})x + n_{\mathrm{B}}(2\kappa + x_0) + 46\log(10) + Q \gtrsim 0$$

$$(4 - n_{\mathrm{B}})x + n_{\mathrm{B}}(2\kappa + x_0) + Q \lesssim 0$$

$$(6 - n_{\mathrm{B}})x + (n_{\mathrm{B}} - 2)\left(2\kappa + x_0 + \frac{Q}{n_{\mathrm{B}} - 4}\right) \lesssim 0$$

$$Q \equiv \Delta N_{\mathrm{tr}}(\beta - 2)(4 - n_{\mathrm{B}})$$

Whittaker-Hill equation

$$\frac{d^2\mathscr{A}}{dz^2} + \left[C + 2q\cos(4z) + 2p\cos(2z)\right]\mathscr{A} = 0$$

$$\Delta(i\mu) \equiv \begin{vmatrix} \ddots & & & & & & & & \\ & \tilde{\zeta}_{-2} & \zeta_{-2} & 1 & \zeta_{-2} & \tilde{\zeta}_{-2} & 0 & 0 & \\ & 0 & \tilde{\zeta}_{0} & \zeta_{0} & 1 & \zeta_{0} & \tilde{\zeta}_{0} & 0 & \\ & 0 & 0 & \tilde{\zeta}_{2} & \zeta_{2} & 1 & \zeta_{2} & \tilde{\zeta}_{2} & \\ & & & & & & & & \ddots \end{vmatrix}$$

$$\zeta_{2n} \equiv \frac{p}{C - (i\mu - 2n)^2}$$

$$\tilde{\zeta}_{2n} \equiv \frac{q}{C - (i\mu - 2n)^2}$$

The formula for the real part of the Floquet exponent is

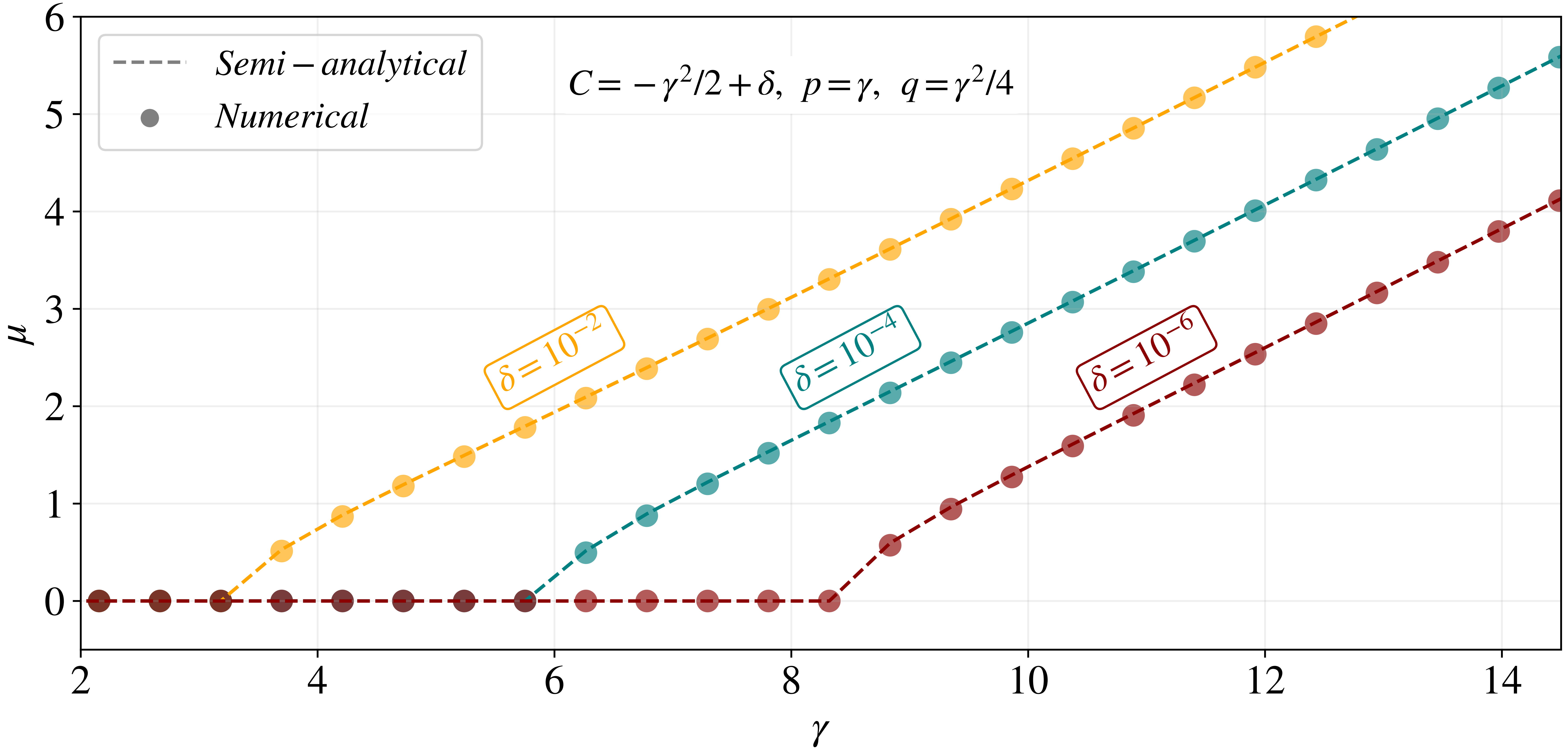
Compute it numerically

$$\mu_R = \frac{1}{\pi}\log\left[\alpha D \pm \sqrt{D^2 - 1}\right]$$

$$D \equiv 2\underline{\Delta(0)}\sinh^2\left(\frac{\pi}{2}\sqrt{|C|}\right) + 1$$

The exponent μ versus γ

$$\mu = max\left\{0{,}0.6\gamma - 0.85\,|\log_{10}\delta|^{0.95}\right\}$$



Dotted colors are from numerical solutions and fit with $e^{\mu z}$

Dash lines are from semi-analytical formula of μ (Floquet analysis) and fit with $e^{\mu z}$