Noether Symmetries in Extended Teleparallel Gauss-Bonnet Cosmology

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Outline of Presentation

- Teleparallel Equivalent of GR.
- Mathematical Formalism.
- Noether Symmetry Approach.
- Outcome of the study.

Teleparallel Equivalent of GR

• To unify electromagnetism and gravitation, the first attempt to modify GR was made by H. Weyl in 1918[1].

• In the late 1920s Einstein himself attempted to unify electromagnetism and gravitation, using the mathematical structure of teleparallelism.

• The TEGR is formulated in terms of the tetrad field and of the corresponding torsion tensor, which is the antisymmetric part of the Weitzenböck connection.

• Even though teleparallel gravity is dynamically completely equivalent to general relativity, it has a very different physical interpretation.

• The modification in the geometrical part leads to several extended theories of gravity such as $f(T)$ gravity [2], $f(T, B)$ gravity [3], $f(T, T_G)$ gravity[4] , $f(T, \phi)$ gravity [5] and so on.

[1] H. Weyl, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*,**1918**, 465 (1918).
[2] N. Tamanini, C. G. Bohmer, *Phys. Rev. D*, **86**, 044009 (2012).
[3] S. Bahamonde, C. G. Bohmer, and M. Wright, Phys. Rev. D, **92**,104042 (2015).
[4] G. Kofinas, E. N. Saridakis, *Phys. Rev. D***90**, 084045 (2014).
[5] M. Gonzalez-Espinoza, G. Otalora, *Eur. Phys. J. C.*, **81**, 480 (2021).

Mathematical Formalism of Teleparallel Gauss Bonnet Gravity:

We consider a flat isotropic and homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) metric.

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\,, \quad (1)$$

Where $a(t)$ is the scale factor and the tetrad field can be described as follow,

$$e^A_\mu = (1, a(t), a(t), a(t))\,, \quad (2)$$

the tetrad e^A_μ (and its inverses E^μ_A) relate to the metric as the fundamental variable of theory through the relations,

$$g_{\mu\nu} = e^A_\mu e^B_\nu \eta_{AB}\,, \qquad \eta_{AB} = E^\mu_A E^\nu_B g_{\mu\nu}\,, \quad (3)$$

The tetrads must satisfy orthogonality conditions which take of the form,

$$e^A_\mu E^\mu_B = \delta^A_B\,, \qquad e^A_\mu E^\nu_A = \delta^\nu_\mu\,, \quad (4)$$

The Weitzenböck connection can be defined as,

$$\Gamma^\sigma_{\nu\mu} := E^\sigma_A \left(\partial_\mu e^A_\nu + \omega^A_{B\mu} e^B_\nu\right)\,, \quad (5)$$

Mathematical Formalism of Teleparallel Gauss Bonnet Gravity:

The torsion scalar can be defined as,

$$T = \frac{1}{4} T^{\alpha}{}_{\mu\nu} T_{\alpha}{}^{\mu\nu} + \frac{1}{2} T^{\alpha}{}_{\mu\nu} T^{\alpha}{}_{\nu\mu} - T^{\alpha}{}_{\mu\alpha} T^{\beta\mu}{}_{\beta}\,, \tag{6}$$

The Gauss-Bonnet term, which has been derived in the TG setting to be defined as,

$$T_G = \left(K_a{}^{i}{}_{e} K_b{}^{ej} K_c{}^{k}{}_{f} K_d{}^{fl} - 2K_a{}^{ij} K_b{}^{k}{}_{e} K_c{}^{e}{}_{f} K_d{}^{fl} + 2K_a{}^{ij} K_b{}^{k}{}_{e} K_f{}^{el} K_d{}^{f}{}_{c} + 2K_a{}^{ij} K_b{}^{k}{}_{e} K_{c,d}^{\ \ el} \right) \delta^{abcd}_{iklk}\,, \tag{7}$$

$$T = 6H^2\,, \quad T_G = 24H^2 \left(\dot{H} + H^2 \right)\,, \tag{8}$$

The action formula can be written as,

$$\mathcal{S}_{f(T,T_G)} = \frac{1}{2\kappa^2} \int \mathrm{d}^4x\, e\, f(T, T_G) + \int \mathrm{d}^4x\, e\mathcal{L}_{\mathrm{m}}\,, \tag{9}$$

The Friedmann equations for this set-up are given by [6],[7].

$$f - 12H^2 f_T - T_G f_{T_G} + 24H^3 \dot{f}_{T_G} = 2\kappa^2 \rho \tag{10}$$

$$f - 4(\dot{H} + 3H^2) f_T - 4H\dot{f}_T - T_G F_{T_G} + \frac{2}{3H} T_G \dot{f}_{T_G} + 8H^2 \ddot{f}_{T_G} = -2\kappa^2 p \tag{11}$$

[6] S. Bahamonde, C. G. Bohmer, *Eur. Phys. J. C*, **76**, 578 (2016).

[7] G. Kofinas, E. N. Saridakis, *Phys. Rev. D*, **90**, 084045 (2014).

Noether Symmetry Approach in $f(T, T_G)$ Gravity:

- The classical Noether symmetry approach was originally established by Emmy Noether [8].
- For every continuous symmetry we can find a corresponding conserved quantity, the Noether symmetry allows one to fix physically interesting cosmological models related to the conserved quantities.
- The existence of Noether symmetries allows to reduce dynamics of the system and then to achieve exact solutions.
- Noether symmetries act as a sort of selection rules to obtain viable models in quantum cosmology.
- The Noether symmetry approach in $f(T, T_G)$ gravity was firstly studied in[9].

[8] E. Noether, *Transp. Theory Statist. Phys*, **90**, 186 (1971).
[9] S. Capozziello, M. D. Laurentis, K. F. Dialektopoulos, *Eur. Phys. J. C*, **76**, 629 (2016).

Noether Symmetry Approach in $f(T, T_G)$ Gravity:

We start with the Euler-Lagrange equations which are given by

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{a}} = \frac{\partial \mathcal{L}}{\partial a}, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{T}} = \frac{\partial \mathcal{L}}{\partial T}, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{T}_G} = \frac{\partial \mathcal{L}}{\partial T_{T_G}}, \tag{12}$$

where the energy condition can be written as

$$E_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial \dot{a}}\dot{a} + \frac{\partial \mathcal{L}}{\partial \dot{T}}\dot{T} + \frac{\partial \mathcal{L}}{\partial \dot{T}_G}\dot{T}_G - \mathcal{L}, \tag{13}$$

Let us consider the canonical variables a, T and T_G in order to derive the $f(T, T_G)$ action through

$$\mathcal{S}_{f(T,T_G)} = \int \mathcal{L}(a, \dot{a}, T, \dot{T}, T_G, \dot{T}_G)\, dt, \tag{14}$$

the action equation in Eq. (9) into its point-like representation using the Lagrange multipliers λ_1 and λ_2 as follows

$$\mathcal{S}_{f(T,T_G)} = 2\pi^2 \int dt \left\{ f(T, T_G)a^3 - \lambda_1 \left[T - 6\left(\frac{\dot{a}}{a}\right)^2 \right] - \lambda_2 \left(T_G - 24\left[\frac{\dot{a}^2\ddot{a}}{a^3}\right] \right) \right\}. \tag{15}$$

Noether Symmetry Approach in $f(T, T_G)$ Gravity:

Thus, the action in Eq. (15) can be written in following form

$$\mathcal{S}_{f(T,T_G)} = 2\pi^2 \int dt \left\{ f(T, T_G) a^3 - a^3 f_T \left[T - 6 \left(\frac{\dot{a}}{a} \right)^2 \right] - a^3 f_{T_G} \left(T_G - 24 \left[\frac{\dot{a}^2 \ddot{a}}{a^3} \right] \right) \right\}, \tag{16}$$

where the point-like Lagrangian is given by

$$\mathcal{L}_{f(T,T_G)} = a^3 \left[f(T, T_G) - T f_T - T_G f_{T_G} \right] + 6a\dot{a}^2 f_T - 8\dot{a}^3 \dot{f}_{T_G} \,. \tag{17}$$

On the other hand, the Euler-Lagrange equations for the variables a, T and T_G can be respectively obtained to give the following

$$6\dot{a}^2 f_T + 12a\ddot{a} f_T + 12a\dot{a} \dot{f}_T - 24 \left[2\dot{a}\ddot{a} \dot{f}_{T_G} + \dot{a}^2 \ddot{f}_{T_G} \right] - 3a^2 f(T, T_G) + 3a^2 T f_T + 3a^2 T_G f_{T_G} = 0 \,, \tag{18}$$

$$\left(-24\dot{a}^2\ddot{a} + a^3 T_G \right) f_{T_G T} + \left(a^3 T - 6a\dot{a}^2 \right) f_{TT} = 0 \,, \tag{19}$$

$$\left(-24\dot{a}^2\ddot{a} + a^3 T_G \right) f_{T_G T_G} + \left(a^3 T - 6a\dot{a}^2 \right) f_{T T_G} = 0 \,. \tag{20}$$

Noether Symmetry Approach in $f(T, T_G)$ Gravity:

Now, the energy condition described in Eq. (13) can be associated with the Lagrangian described in Eq. (17) to give

$$\left(12a\dot{a}\, f_T - 24\dot{a}^2 \dot{f}_{T_G}\right)\dot{a} - a^3 f(T, T_G) + a^3 T_G\, f_{T_G} = 0\,. \tag{21}$$

The Lagrangian $\mathcal{L}$ admits the Noether symmetry if its Lie derivative along a vector field X vanishes, meaning that

$$\mathcal{L}_X \mathcal{L} = 0 \implies X\mathcal{L} = 0\,. \tag{22}$$

The system with Lagrangian $\mathcal{L} = \mathcal{L}(t, q, \dot{q})$, the action is said to invariant under infinitesimal transformations if the Rund-Trautman identity [10] holds,

$$\chi^{(1)}\mathcal{L} + \frac{d\xi}{dt}\mathcal{L} = \frac{dg}{dt} \tag{23}$$

where,

$$\chi^{(1)} = \chi + (\dot{\eta}^i - \dot{q}^i\dot{\xi})\frac{\partial}{\partial \dot{q}^i}, \quad qi = a, T, T_G \tag{24}$$

Together with the independent variable of cosmic time t, the generator vector can be described as

$$\chi = \xi(t, a, T, T_G)\partial_t + \sum \eta_{qi}(t, a, T, T_G)\partial_{qi}\,. \tag{25}$$

[10] S. Basilakos, M. Tsamparlis, and A. Paliathanasis, *Phys. Rev. D*, **83**, 103512 (2011).

Noether Symmetry Approach in $f(T, T_G)$ Gravity:

The system derived from the condition in Eq. (23) consist partial differential equations as,

$$-8f_{T_G,T}\eta_{T,t} - 8\eta_{T_G,t}f_{T_G,T_G} - 6af_T\xi_{,a} = 0 \tag{26}$$

$$-12a\xi_{,T} - 24\eta_{a,t}f_{T_G,T} + 6af_T\xi_{,T} = 0 \tag{27}$$

$$-8f_{T_G,T}\eta_{T,a} - 8f_{T_G,T_G}\eta_{T_G,a} = 0 \tag{28}$$

$$24\xi_{,T}f_{T_G,T_G} + 24\xi_{,T_G}f_{T_G,T} = 0 \tag{29}$$

$$-24\eta_{a,T}f_{T_G,T_G} - 24\eta_{a,T_G}f_{T_G,T} = 0 \tag{30}$$

$$-6af_T\xi_{,T_G} - 24\eta_{a,t}f_{T_G,T_G} = 0 \tag{31}$$

$$12af_T\eta_{a,T} = 0 \tag{32}$$

$$12af_T\eta_{a,T_G} = 0 \tag{33}$$

$$-24\eta_{a,T}f_{T_G,T} = 0 \tag{34}$$

$$-24\eta_{a,T_G}f_{T_G,T_G} = 0 \tag{35}$$

$$24\xi_{,T_G}f_{T_G,T_G} = 0 \tag{36}$$

$$24\xi_{,a}f_{T_G,T} = 0 \tag{37}$$

$$24\xi_{,a}f_{T_G,T_G} = 0 \tag{38}$$

$$24\xi_{,T}f_{T_G,T} = 0 \tag{39}$$

$$\eta_{a,t}12af_T + a^3f\xi_{,a} - a^3Tf_T\xi_{,a} - a^3T_Gf_{T_G}\xi_{,a} = g_{,a} \tag{40}$$

$$a^3f\xi_{,T} - a^3Tf_T\xi_{,T} - a^3T_Gf_{T_G}\xi_{,T} = g_{,T} \tag{41}$$

Noether Symmetry Approach in $f(T, T_G)$ Gravity:

$$a^3 f\xi_{,a} - a^3 T f_T \xi_{,T_G} - a^3 T_G f_{T_G} \xi_{,T_G} = g_{,T_G} \tag{42}$$

$$\eta_a 6 f_T + \eta_T 6 a f_{T,T} + \eta_{T_G} 6 a f_{T,T_G} - 6 a f_T \xi_{,t} + 12 a f_T \eta_{a,a} = 0 \tag{43}$$

$$-\eta_T 8 f_{T_G,T,T_G} - \eta_{T_G} 8 f_{T_G,T_G,T_G} - 24\eta_{a,a} f_{T_G,T_G} + 24\xi_{,t} f_{T_G,T_G} - 8 f_{T_G,T}\eta_{T,T_G} - 8 f_{T_G,T_G}\eta_{T_G,T_G} = 0 \tag{44}$$

$$-\eta_T 8 f_{T_G,T,T} - \eta_{T_G} 8 f_{T_G,T_G,T} - 24\eta_{a,a} f_{T_G,T} + 24\xi_{,t} f_{T_G,T} - 8 f_{T_G,T}\eta_{T,T} - 8 f_{T_G,T_G}\eta_{T_G,T} = 0 \tag{45}$$

$$3a^2\eta_a f - 3a^2\eta_a T f_T - \eta_a 3a^2 T_G f_{T_G} - \eta_T a^3 T f_{TT} - \eta_T a^3 T_G f_{T_G,T} + a^3 f \xi_{,t} - a^3 T f_T \xi_{,t} - a^3 T f_{T,T_G}\eta_{T_G} - a^3 T_G f_{T_G}\xi_{,t} - \eta_{T_G} a^3 T_G f_{T_G,T_G} = g,t \tag{46}$$

The case $f(T, T_G) = b_0 T_G^k + t_0 T^m$

In this case, we substitute this form of $f(T, T_G)$[11] in the system (26-46) for the particular case $k = 1$ and arbitrary m. The Noether symmetry vector assumes the form as follow,

$$\chi = \left(\xi = \tau_1, \quad \eta_a = \alpha_0 a^{1-\frac{3}{2m}}, \quad \eta_{T_G} = \eta_{T_G}(t, a, T, T_G), \quad \eta_T = -\frac{3\alpha_0 T a^{-\frac{3}{2m}}}{m}, \quad g = \tau_2\right). \tag{47}$$

The pointlike Lagrangian presented in Eq. (17) takes the form as,

$$\mathcal{L} = a^3(1-m)T^m t_0 + 6a\dot{a}^2 t_0 m T^{m-1}. \tag{48}$$

The Euler-Lagrange equation for T_G may give the Lagrange multiplier T_G and the Euler-Lagrange equation for variable a in Eq. (12) and the energy condition equation in Eq.(13) respectively can be written as,

$$12t_0 m\left(\dot{a}^2 T^{m-1} + a\ddot{a}T^{m-1} + a\dot{a}(m-1)T^{m-2}\dot{T}\right) - 3a^2(1-m)T^m t_0 - 6\dot{a}^2 t_0 m T^{m-1} = 0, \tag{49}$$

$$6a\dot{a}^2 t_0 m T^{m-1} - a^3(1-m)T^m t_0 = 0. \tag{50}$$

[11] S. Capozziello, M. D. Laurentis, K. F. Dialektopoulos, *Eur. Phys. J. C*, **76**, 629 (2016).

Using Eq. (8), the above equations in Eqs. (49–50) can be reformulated to take the form

$$6^m(2m-1)t_0\left(\frac{\dot{a}^2}{a^2}\right)^{m-1}\left((3-2m)\dot{a}^2+2ma\ddot{a}\right)=0\,, \quad (51)$$

$$6^m(2m-1)t_0a^3\left(\frac{\dot{a}^2}{a^2}\right)^{m}=0\,, \quad (52)$$

The differential equation Eq. (51) admit the following solution

$$a(t)=\tilde{c}_2(3t-2\tilde{c}_1m)^{\frac{2m}{3}}\,, \quad \text{for} \quad m\neq\frac{1}{2}\,, \quad (53)$$

where $\tilde{c}_1, \tilde{c}_2$ are the integrating constants.
The deceleration parameter takes the form $-1+\frac{3}{2m}$.
Hence this exact solution can explain the accelerating universe for the parameter m, within the range $m<0 \quad \text{or} \quad m>\frac{3}{2}$.
Moreover, these cosmology expressions admit de-Sitter solution as follow,

$$a(t)=e^{st}\,, \quad \text{where} \quad \left(s\neq0\wedge\left(m=\frac{1}{2}\vee t_0=0\right)\right)\vee(\Re(m)>0\wedge s=0)\,. \quad (54)$$

Here s, t_0 are the real constants and $\Re(m)$ represents real part of m.

The case $f(T, T_G) = -T + je^{mT_G}$

In this form of $f(T, T_G)$ [12], the nontrivial Noether vector takes the form,

$$\chi = \left(\xi = k_1 t + k_2, \eta_a = \frac{ak_1}{3}, \eta_{T_G} = \frac{2k_1}{m} + c_1 e^{-mT_G}, \eta_T = \eta_T(t, a, T, T_G), g = \tau_2\right). \tag{55}$$

In this case the Lagrangian Eq. (17) takes the form

$$\mathcal{L} = -6a\dot{a}^2 + je^{mT_G}\left(a^3 - T_G a^3 m - 8\dot{a}^3 \dot{T}_G m^2\right), \tag{56}$$

Using the Euler-Lagrange equations in Eq. (12) for the variable a and the energy condition in Eq. (13), respectively give,

$$-6\dot{a}^2 - 12a\ddot{a} - 24jm^2 e^{mT_G}\left(m\dot{T}_G\dot{a}^2\dot{T}_G + 2\dot{a}\ddot{a}\dot{T}_G + \dot{a}^2\ddot{T}_G\right) - je^{mT_G}\left(3a^2 - 3ma^2 T_G\right) = 0, \tag{57}$$

$$-6a\dot{a}^2 - je^{mT_G}\left(24\dot{a}^3\dot{T}_G m^2 + a^3 - T_G a^3 m\right) = 0. \tag{58}$$

The cosmological solution in this case can then be obtained to give exact solution as,

$$a(t) = e^{\lambda t} \quad \text{where} \quad 24\lambda^4 m - 1 \neq 0 \quad \text{and} \quad j = \frac{6\lambda^2 e^{-24\lambda^4 m}}{24\lambda^4 m - 1}, \tag{59}$$

[12] B. Mirza and F. Oboudiat, *J. Cosmo. Astropar. Phys.*, **2017**, 011 (2017).

The case $f(T, T_G) = -T + je^{m\sqrt{T_G}}$

In this form of $f(T, T_G)$[13], the coefficients of Noether vector takes the form as follow,

$$\chi = \left(\xi = k_1 t + k_2, \quad g = \tau_2, \quad \eta_{T_G} = -\frac{4k_1 T_G}{1 - m\sqrt{T_G}} + \frac{c_1 (T_G)^{3/2} e^{-m\sqrt{T_G}}}{1 - m\sqrt{T_G}}, \right.$$
$$\left. \eta_T = \eta_T(t, a, T, T_G), \quad \eta_a = \frac{ak_1}{3}\right),$$

In this case the Lagrangian Eq.(17) takes the form as follow,

$$\mathcal{L} = -6a\dot{a}^2 + e^{m\sqrt{T_G}}\left(a^3 j - \frac{\sqrt{T_G} jma^3}{2} + \frac{2\dot{a}^3 \dot{T}_G jm}{(T_G)^{\frac{3}{2}}} - \frac{2(\dot{a})^3 jm^2 \dot{T}_G}{T_G}\right), \tag{60}$$

The Euler-Lagrange equations from Eq. (12) and Energy condition from Eq. (13) respectively can be expressed as

$$\mu(1 - \sqrt{T_G} m)e^{m\sqrt{T_G}} + e^{m\sqrt{T_G}}\left(-\frac{3\dot{a}^2(\dot{T}_G)^2 jm^2}{(T_G)^2} - 3a^2 j + \frac{3a^2\sqrt{T_G} jm}{2}\right)$$
$$-6\dot{a}^2 - 12a\ddot{a} = 0. \tag{61}$$

[13] E. V. Linder, *Phys.Rev.D*, **201782**, 109902 (2010).

$$\text{where } \mu = \left((T_G)^{\frac{-3}{2}} \left(\frac{3m^2 j(\dot{T}_G)^2 \dot{a}^2}{\sqrt{T_G}} + 12\dot{a}\ddot{a}\dot{T}_G jm + 6\dot{a}^2 \ddot{T}_G jm \right) - 9\dot{a}^2 (\dot{T}_G)^2 jm (T_G)^{\frac{-5}{2}} \right),$$

$$-6a\dot{a}^2 + e^{m\sqrt{T_G}} \left(\frac{6\dot{a}^3 jm\dot{T}_G}{T_G} \left(\frac{1 - m\sqrt{T_G}}{\sqrt{T_G}} \right) - a^3 j + \frac{\sqrt{T_G} jma^3}{2} \right) = 0 . \tag{62}$$

The cosmological solution in this case can be obtained as follow,

$$a(t) = e^{\zeta t} \quad \text{where} \quad \sqrt{6} - 6\zeta^2 m \neq 0 \quad \text{and} \quad j = -\frac{6\sqrt{6}\zeta^2 e^{-2\sqrt{6}\zeta^2 m}}{\sqrt{6} - 6\zeta^2 m}, \tag{63}$$

Outcome of Study:

- In this work, we explore $f(T, T_G)$ gravity through the prism of Noether symmetries.
- We consider the Rund-Trautman identity presented in Eq.(23) by which we have obtained the system of differential equations.
- For the case $f(T, T_G) = b_0 T_G + t_0 T^m$, the non-trivial Noether vector form has been obtained and the exact solutions has been discussed.
- Similarly the other two cases has been studied to analyse the systematic approach to find the exact solutions.
- The corresponding deceleration parameter has been obtained for the exact solutions which will help to study the late time cosmic acceleration phenomena.

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Thank you