Dynamical Friction in Fuzzy Dark Matter Circular Orbits

Cosmology from Home 2023

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Axions

Properties

- Scalar particles
- Possible mass range $m_a \approx 10^{-22} - 10^{-6}$ eV
- Weak interaction with rest of standard model

Motivation: Particle Physics

Standard model

- Solves strong CP problem [Peccei and Quinn, 1977]
- Mass around $10^{-10} \lesssim m_a \lesssim 10^{-3}$ eV
- Referred to as "QCD-axion" (QCDA)

Motivation: Particle Physics

Standard model

- Solves strong CP problem [Peccei and Quinn, 1977]
- Mass around $10^{-10} \lesssim m_a \lesssim 10^{-3}$ eV
- Referred to as "QCD-axion" (QCDA)

String theory

- Generated by compacted spacial dimension [Svrcek and Witten, 2006]
- No limit on mass, extrapolate to $10^{-21} \lesssim m_a \lesssim 10^{-18}$ eV
- Referred to as "Ultra Light Axions" (ULA)

Motivation: Cosmology

Possible Dark Matter candidate

- Very weak interaction with standard model
- Low mass $\rightarrow$ produced abundantly during early universe
- Mimics CDM on large scales
- But solves problems with CDM on small scales:
  - Cusp-core problem [Hu et al., 2000]
  - Core stalling (e.g. Fornax dwarf galaxy) [Lora et al., 2012]
  - Missing satellite problem [Marsh and Silk, 2013]
  - etc.

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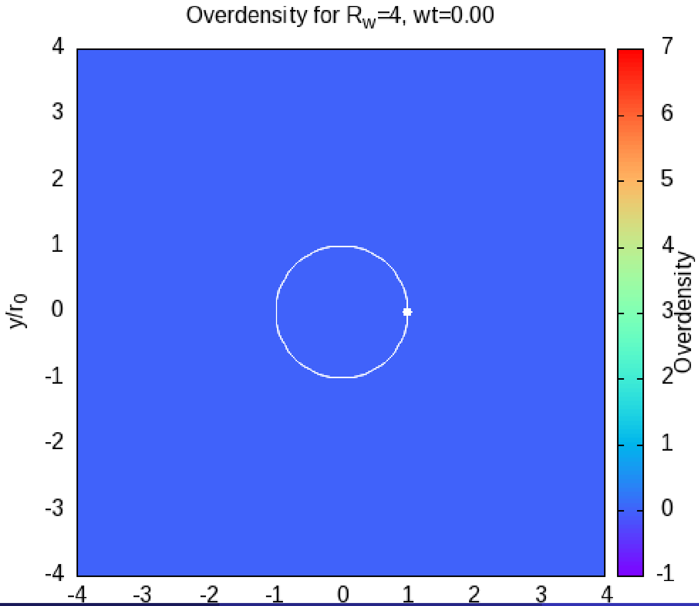
Density wake



Introduction

What is Dynamical Friction?

- A perturber (Planet, Black Hole, etc.) moves through continous homogeneous medium
- Due to gravitational pull it will start dragging material behind id
- The wake will exert a gravitational pull back onto the perturber

Brief History

- [Chandrasekhar, 1943] for the DF force by an homogeneous and isotropic distribution of stars
- [Ostriker, 1999] extended to other media using linear response theory
- [Hui et al., 2017] considered linear motion in FDM

Schrödinger equation

$$i\partial_t \psi = -\frac{\hbar}{2m_a}\Delta_{\mathbf{r}}\psi + \frac{m_a}{\hbar}\Phi\,\psi$$

$$\Delta_{\mathbf{r}}\Phi_0 = 4\pi G\rho\ , \qquad \rho \equiv |\psi|^2$$

$$\Delta_{\mathbf{r}}\Phi_p = 4\pi G M h(t)\delta^D(\mathbf{r} - \mathbf{r}_p(t))$$

with $\mathbf{r}_p(t) = (r_0\cos(\Omega t),\ r_0\sin(\Omega t),\ 0)$

Madelung formulation

Madelung approach

[Madelung, 1927]

$$\psi = \sqrt{\rho}\,\mathrm{e}^{i\theta}$$

$$\mathbf{v} = \frac{\hbar}{m_a}\nabla_{\mathbf{r}}\theta$$

Madelung formulation

Madelung approach

[Madelung, 1927]

$$\psi = \sqrt{\rho}\,\mathrm{e}^{i\theta}$$

$$\mathbf{v} = \frac{\hbar}{m_a}\nabla_{\mathbf{r}}\theta$$

Perturbation

$$\rho \to \bar{\rho} + \delta\rho \quad \alpha(\mathbf{r}, t) \equiv \frac{\rho(\mathbf{r},t)}{\bar{\rho}} - 1$$

$$\mathbf{v} \to \bar{\mathbf{v}} + \delta\mathbf{v}$$

Madelung formulation

Madelung approach

[Madelung, 1927]

$$\psi = \sqrt{\rho}\,\mathrm{e}^{i\theta}$$

$$\mathbf{v} = \frac{\hbar}{m_a}\nabla_{\mathbf{r}}\theta$$

Perturbation

$$\rho \to \bar{\rho} + \delta\rho \quad \alpha(\mathbf{r}, t) \equiv \frac{\rho(\mathbf{r},t)}{\bar{\rho}} - 1$$

$$\mathbf{v} \to \bar{\mathbf{v}} + \delta\mathbf{v}$$

$\bar{\mathbf{v}} = 0$ and $\Phi_0 = 0$

$$\partial_t^2\alpha + \frac{\hbar^2}{4m_a^2}\Delta_{\mathbf{r}}^2\alpha = -\Delta_{\mathbf{r}}\Phi_p$$

Green's function

Fourier space

Used for calculation of DF

$$G_{\text{ret}}(\mathbf{r},\tau) = \frac{1}{\Omega r_0^3} \lim_{\epsilon\to 0^+} \int_{\tilde{\mathbf{k}}} \int_{\tilde{\omega}} \frac{e^{i(\tilde{\mathbf{k}}\cdot\tilde{\mathbf{r}}-\tilde{\omega}\tilde{\tau})}}{\tilde{k}^4/R_\Omega^2 - (\tilde{\omega}+i\epsilon)^2}$$

with dimensionless radius $R_\Omega = \frac{m_a \Omega r_0^2}{\hbar} = \frac{2r_0}{\lambda_{DB}}$

Green's function

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Used for calculation of DF

$$G_{\mathrm{ret}}(\mathbf{r},\tau) = \frac{1}{\Omega r_0^3} \lim_{\epsilon\to 0^+} \int_{\tilde{\mathbf{k}}} \int_{\tilde{\omega}} \frac{e^{i(\tilde{\mathbf{k}}\cdot\tilde{\mathbf{r}}-\tilde{\omega}\tilde{\tau})}}{\tilde{k}^4/R_\Omega^2 - (\tilde{\omega}+i\epsilon)^2}$$

with dimensionless radius $R_\Omega = \frac{m_a \Omega r_0^2}{\hbar} = \frac{2r_0}{\lambda_{DB}}$

Overdensity

$$\alpha(\mathbf{r},t) = 4\pi G M \int \mathrm{d}t' G_{\mathrm{ret}}(\mathbf{r}-\mathbf{r}_p(t'), t-t')$$

Dynamical Friction

Newton's Equation

$$\mathbf{F}_{\mathrm{DF}}(t) = GM\bar{\rho}\int \mathrm{d}u^3 \frac{\mathbf{r}_p(t) - \mathbf{u}}{|\mathbf{r}_p(t) - \mathbf{u}|^3}\alpha(\mathbf{u}, t)$$

Dynamical Friction

Newton's Equation

$$\begin{aligned}\mathbf{F}_{\mathrm{DF}}(t) &= GM\bar{\rho}\int \mathrm{d}u^3\,\frac{\mathbf{r}_p(t)-\mathbf{u}}{|\mathbf{r}_p(t)-\mathbf{u}|^3}\alpha(\mathbf{u},t)\\ &= 4\pi(GM)^2\bar{\rho}\lim_{\epsilon\to 0^+}\int \mathrm{d}u^3\int \mathrm{d}t'\int_{\tilde{\mathbf{k}}}\int_{\tilde{\omega}}\\ &\cdot\frac{\mathbf{r}_p(t)-\mathbf{u}}{|\mathbf{r}_p(t)-\mathbf{u}|^3}\frac{e^{i(\tilde{\mathbf{k}}\cdot(\mathbf{u}-\mathbf{r}_p(t'))-\tilde{\omega}(t-t')}}{\tilde{k}^4/R_\Omega^2-(\tilde{\omega}+i\epsilon)^2}\end{aligned}$$

Dynamical Friction

Newton's Equation

$$\mathbf{F}_{\mathrm{DF}}(t) = GM\bar{\rho}\int \mathrm{d}u^3 \frac{\mathbf{r}_p(t)-\mathbf{u}}{|\mathbf{r}_p(t)-\mathbf{u}|^3}\alpha(\mathbf{u},t)$$
$$= 4\pi(GM)^2\bar{\rho}\lim_{\epsilon\to 0^+}\int \mathrm{d}u^3\int \mathrm{d}t'\int_{\tilde{\mathbf{k}}}\int_{\tilde{\omega}}$$
$$\cdot\frac{\mathbf{r}_p(t)-\mathbf{u}}{|\mathbf{r}_p(t)-\mathbf{u}|^3}\frac{e^{i(\tilde{\mathbf{k}}\cdot(\mathbf{u}-\mathbf{r}_p(t'))-\tilde{\omega}(t-t')}}{\tilde{k}^4/R_\Omega^2-(\tilde{\omega}+i\epsilon)^2}$$

Using

- Fourier transform of Coloumb potential $\int \mathrm{d}u^3\frac{\mathbf{u}}{u^3}e^{i\mathbf{uk}} = 4\pi i\frac{\mathbf{k}}{k^2}$
- Reyleigh expansion for exponent $e^{i\mathbf{kr}} = \sum_{l,m} = Y_l^m(\hat{\mathbf{k}})Y_l^m(\hat{\mathbf{r}})j_l(kr)$
- Helicity decomposition

Dynamical Friction: Results

Complex friction

$$I = \sum_{l=1}^{\infty} \sum_{m=-l}^{l-2} (-1)^m \frac{(l-m)!}{(l-m-2)!} \frac{S_{l,l-1}(m,R_\Omega,t) - S^*_{l,l-1}(m+1,R_\Omega,t)}{\Gamma(\frac{1-l-m}{2})\Gamma(1+\frac{l-m}{2})\Gamma(\frac{3-l+m}{2})\Gamma(1+\frac{l+m}{2})}$$

Scattering amplitude

$$S_{l,l-1}(m, R_\Omega, t) =$$
$$\lim_{\epsilon\to 0^+} \int_{\tilde\omega} \int_{-\infty}^{\infty} \mathrm{d}\tilde\tau \; h(\tilde t - \tilde\tau)\, e^{i(m-\tilde\omega)\tilde\tau} \int_0^\infty d\tilde k \; \frac{\tilde k\, j_l(\tilde k)\, j_{l-1}(\tilde k)}{\tilde k^4/R_\Omega^2 - (\tilde\omega + i\epsilon)^2}$$

Force

$$\mathbf{F}_{\mathrm{DF}}(t) = -4\pi \left(\frac{GM}{\Omega r_0}\right)^2 \rho \left(\Re(I)\,\hat{\mathbf{r}}(t) + \Im(I)\,\hat{\boldsymbol{\varphi}}(t)\right)$$

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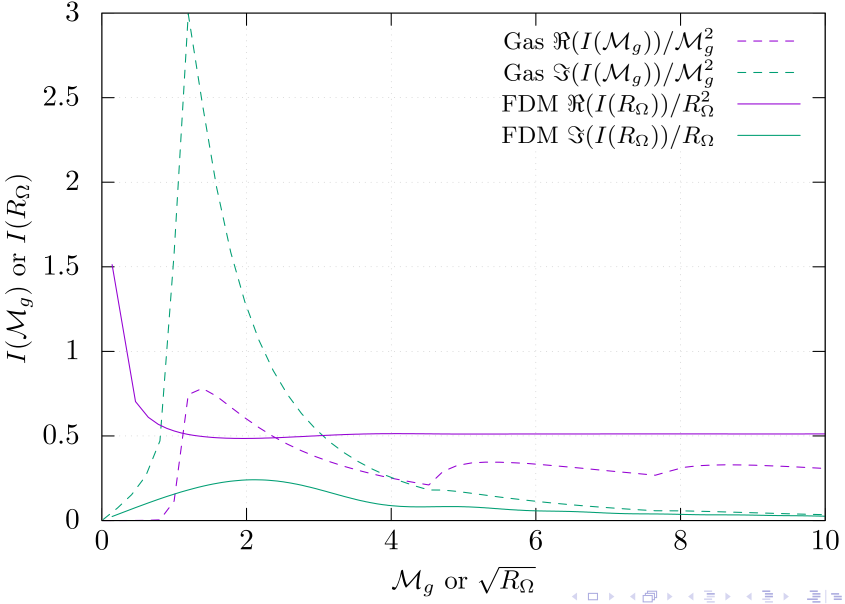
Steady state

Properties

- Perturber is rotating for infinite time ($h(t) = 1$)
- Fully analytic solution
- Multipole expansion converges quickly
- Displays an infrared divergency

Steady state: Comparison to gas

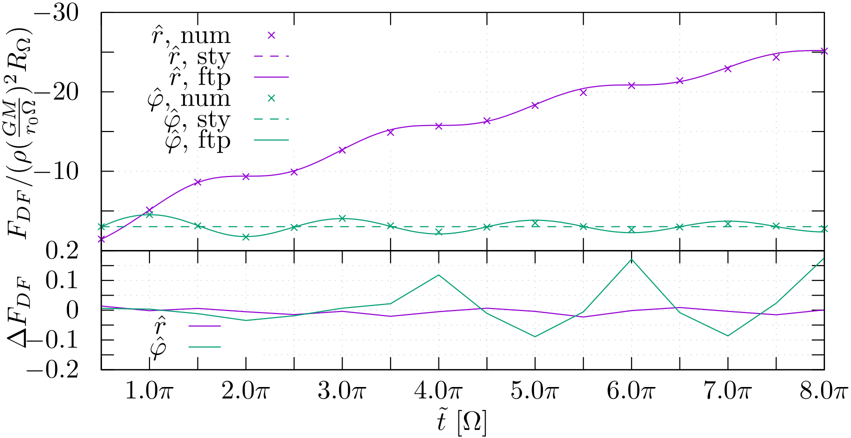


Finite time

Properties

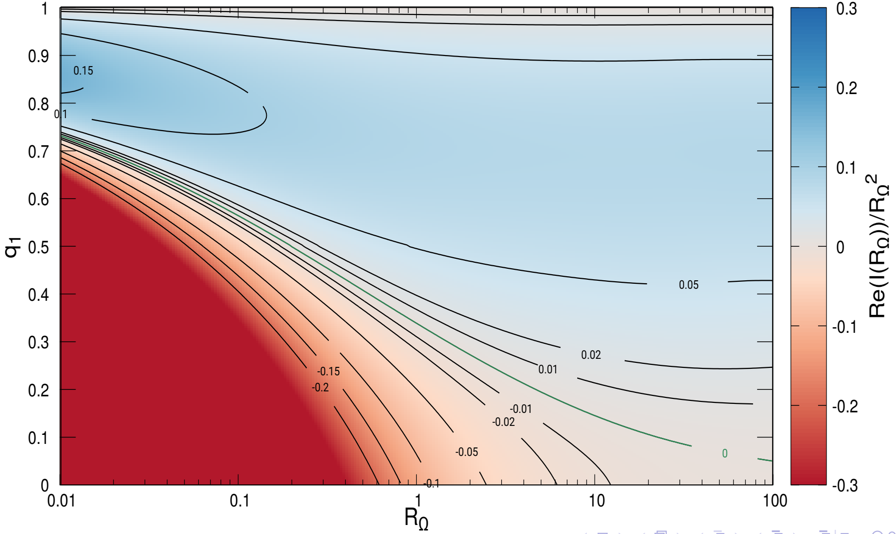
- Perturber starts rotating at $t = 0$ ($h(t > 0) = 1$)
- No analytical solution
- Displays an infrared divergency
- Does not converge to steady state within finite time

Finite time

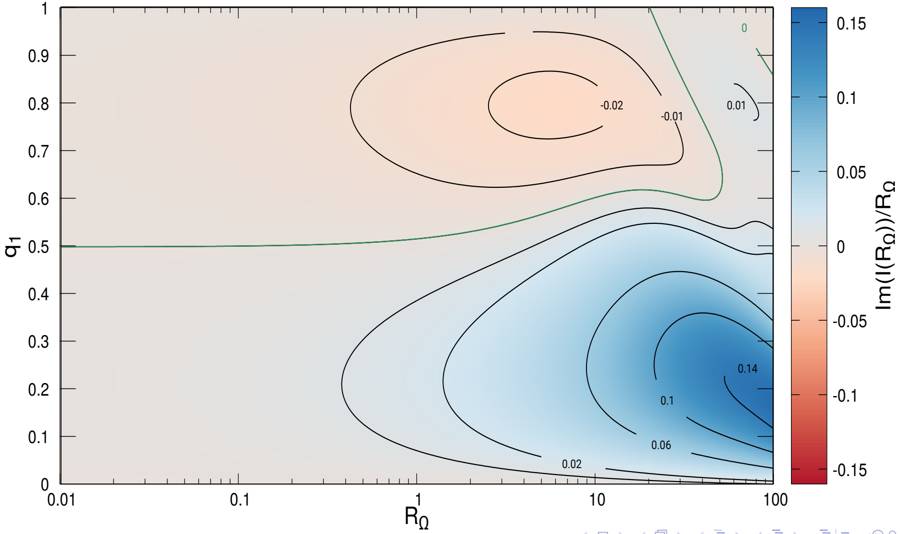


"num": numerical with velocity dispersion ; "sty": steady state ; "ftp": finite time

Binary: Steady State



Binary: Steady State

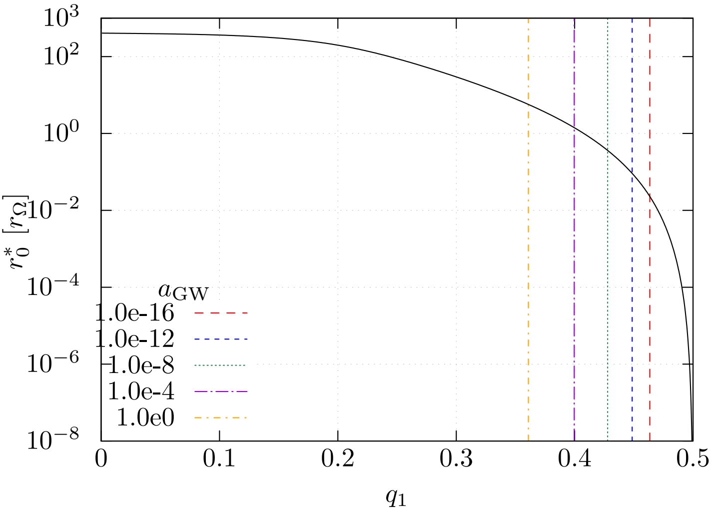


Orbital Stagnation

Orbital stagnation

- DF leads to orbital decay
- DF for binaries can switch sign $\rightarrow$ stable radii
- Use loss of angular momentum
  $\Rightarrow$ System exhibits stable radii

Orbital stagnation



$$r_\Omega \equiv \frac{1}{GM}\left(\frac{\hbar}{2m_a}\right)^2 \; ; \; a_{\rm GW} \equiv \sqrt{\frac{G^5\mu^3 M^4}{c^{10} r_\Omega^{11} \rho^2}}$$

Summary

Summary

- Axions are well motivated DM candidate
- Fully analytical steady state solution
- Infrared divergence
- No exact time-convergence
- DF for binaries can switch sign ⇒ Stable orbits

Summary

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- Axions are well motivated DM candidate
- Fully analytical steady state solution
- Infrared divergence
- No exact time-convergence
- DF for binaries can switch sign $\Rightarrow$ Stable orbits

Thank you for your attention!

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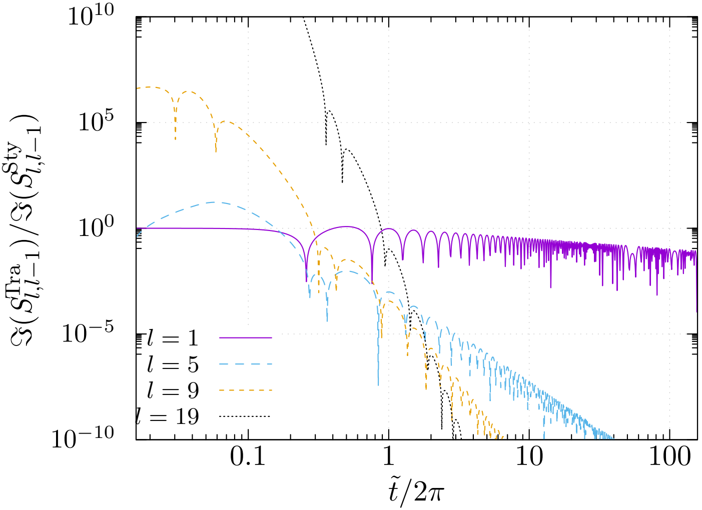
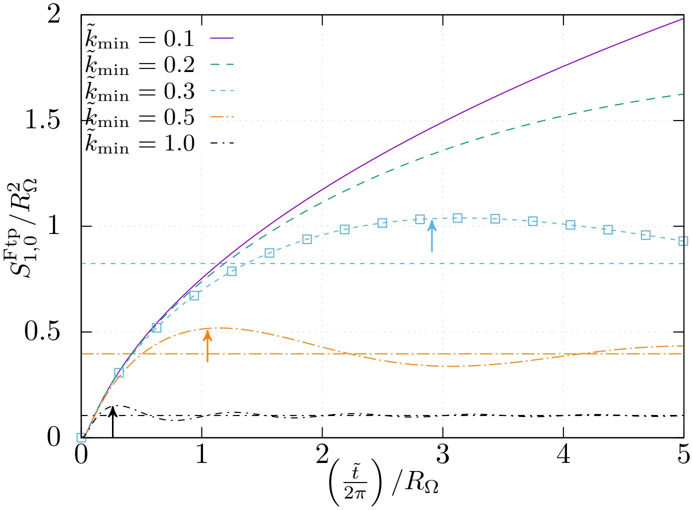
Appendix

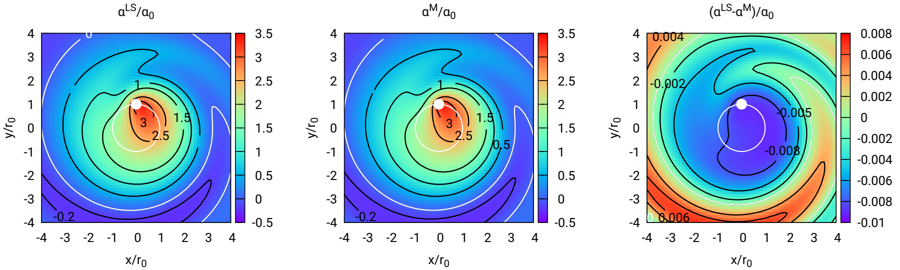
Finite time: Convergence



Finite time: Divergence

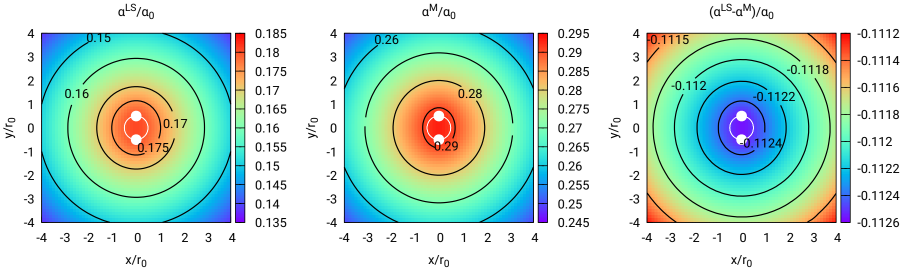


Comparison Velocity dispersion



$$R_\Omega = 4 \; ; \; R_\sigma = 0.1$$

Comparison Velocity dispersion



$$R_\Omega = 0.017 \; ; \; R_\sigma = 0.1$$

Orbital decay

			CDM		FDM linear		FDM circular		
n	r_0	M	C	τ	C	τ	R_Ω	$\Im(I)$	τ
1	7.60	0.37	4.29	112	2.46	215	17.8	1.46	362
2	1.05	1.82	3.32	9.7	1.88	12	10.08	1.64	14
3	0.43	3.63	2.45	0.62	0.29	2.2	1.94	0.39	1.63
4	0.24	1.32	2.50	0.37	0.033	10	0.62	0.078	4.23
5	7.79	1.76	3.46	21.3	2.32	31	15.58	1.41	51

Column "CDM" and "FDM linear" taken from [Hui et al., 2017]

r_0 in [Kpc] ; M in $[10^5 M_\odot]$; τ in [Gyr]

DF on circular orbits

Single perturber: Steady state

Steady state

$$h(t) = 1$$

$$\int_{\tilde{\omega}} h(\tilde{t} - \tilde{\tau}) e^{i(m-\tilde{\omega})\tilde{\tau}} = 2\pi\delta^D(m - \tilde{\omega})$$

$l \neq 1$ and $m \neq 0$

$$S^{\mathrm{Sty}}_{l,l-1}(m, R_\Omega) = \frac{i\pi R_\Omega}{4m}\Big[j_l(\sqrt{mR_\Omega})\, h^{(1)}_{l-1}(\sqrt{mR_\Omega}) + \frac{i}{\sqrt{mR_\Omega}} I_{l+1/2}(\sqrt{mR_\Omega})\, K_{l-1/2}(\sqrt{mR_\Omega})\Big]$$

Single perturber: Steady state

$l \neq 1$ and $m = 0$

$$S^{\mathrm{Sty}}_{l,l-1}(0, R_\Omega) = \frac{3\pi R_\Omega^2}{18 - 80l^2 + 32l^4}$$

$l = 1$ and $m = 0$

$$\begin{aligned}
S^{\mathrm{Sty}}_{1,0}(0, R_\Omega) &= R_\Omega^2 \int_{\tilde{k}_{\min}}^{\infty} \frac{d\tilde{k}}{\tilde{k}^3} j_1(\tilde{k}) j_0(\tilde{k}) \\
&= \frac{R_\Omega^2}{40\tilde{k}_{\min}^5} \Big[4 - 4\pi\tilde{k}_{\min}^5 + \left(4\tilde{k}_{\min}^4 - 2\tilde{k}_{\min}^2 - 4\right)\cos(2\tilde{k}_{\min}) \\
&\qquad + \tilde{k}_{\min}\left(2\tilde{k}_{\min}^2 - 3\right)\sin(2\tilde{k}_{\min}) + 8\tilde{k}_{\min}^5 \mathrm{Si}(2\tilde{k}_{\min}) \Big]
\end{aligned}$$

Single perturber: Finite time

Finite time

$$h(\tau) = \begin{cases} 1 & \tau \leq t \\ 0 & \tau > t \end{cases}$$

$$\int_{\tilde{\omega}} h(\tilde{t} - \tilde{\tau}) e^{i(m-\tilde{\omega})\tilde{\tau}} = \lim_{\eta \to 0^+} \frac{e^{i(m-\tilde{\omega})\tilde{t}}}{i\big(m - \tilde{\omega} - i\eta\big)}$$

For all l, m

$$S^{\text{Ftp}}_{l,l-1}(m, R_\Omega, t) = S^{\text{Sty}}_{l,l-1}(m, R_\Omega) + S^{\text{Tra}}_{l,l-1}(m, R_\Omega, t)$$

$$S^{\text{Tra}}_{l,l-1}(m, R_\Omega, t) = -\frac{R_\Omega}{2} e^{im\tilde{t}} \int_0^\infty \frac{d\tilde{k}}{\tilde{k}} j_l(\tilde{k}) j_{l-1}(\tilde{k}) \cdot \left(\frac{e^{-i(\tilde{k}^2/R_\Omega - i\epsilon)\tilde{t}}}{\tilde{k}^2/R_\Omega - m - i\epsilon} + \frac{e^{i(\tilde{k}^2/R_\Omega + i\epsilon)\tilde{t}}}{k^2/R_\Omega + m + i\epsilon} \right)$$

Single perturber: Finite time

$l \neq 1$ and $m \neq 0$

$$S^{\mathrm{Tra}}_{l,l-1} = -\frac{R_\Omega}{2} e^{im\tilde{t}} \int_0^\infty \frac{\mathrm{d}\tilde{\chi}}{\tilde{\chi}} \big[j_l((1+i)\tilde{\chi}) j_{l-1}((1+i)\tilde{\chi}) - j_l((1-i)\tilde{\chi}) j_{l-1}((1-i)\tilde{\chi}) \big] \cdot \frac{e^{-2\tilde{t}\tilde{\chi}^2/R_\Omega}}{2i\tilde{\chi}^2/R_\Omega + m + i\epsilon}$$

Binary perturber: Steady state

$l \neq 1$ and $m \neq 0$

$$S^{a,b}_{l,l-1}(m, R_\Omega) =$$

$$\frac{\pi R_\Omega}{4} \begin{cases} \frac{i}{m}\Big[h^{(1)}_l(q_a\sqrt{mR_\Omega})j_{l-1}(q_b\sqrt{mR_\Omega}) \\ -\frac{i\ \mathrm{K}_{l+1/2}(q_a\sqrt{mR_\Omega})\mathrm{I}_{l-1/2}(q_b\sqrt{mR_\Omega})}{\sqrt{q_a q_b m R_\Omega}} + \frac{q_b^{l-1}}{q_a^{l+1}}\frac{2i}{|m|R_\Omega}\Big] & (q_a > q_b) \\ \frac{i}{m}\Big[j_l(q_a\sqrt{mR_\Omega})h^{(1)}_{l-1}(q_b\sqrt{mR_\Omega}) \\ +\frac{i\ \mathrm{I}_{l+1/2}(q_a\sqrt{mR_\Omega})\mathrm{K}_{l-1/2}(q_b\sqrt{mR_\Omega})}{\sqrt{q_a q_b m R_\Omega}}\Big] & (q_a < q_b) \end{cases}$$

Binary perturber: Steady state

$l \neq 1$ and $m = 0$

$$S^{a,b}_{l,l-1}(m, R_\Omega) =$$

$$\frac{\pi R_\Omega}{4} \begin{cases} R_\Omega \left(\frac{q_b}{q_a}\right)^{l-1} \frac{4(3+4l(2+l))q_a^4 + 2(9-4l^2)q_a^2 q_b^2 + (3+4l(l-2))q_b^4}{(9-40l^2+16l^4)q_a^2} & (q_a > q_b) \\ -R_\Omega \left(\frac{q_a}{q_b}\right)^{l} \frac{(2l-3)q_a^2 - (2l+3)q_b^2}{9-40l^2+16l^4} & (q_a < q_b) \end{cases}$$

Binary perturber: Steady state

$l = 1$, $m = 0$ and $q_1 > q_2$

$$
\begin{aligned}
S^{a,b}_{l,l-1}(0,R_\Omega) = \frac{R_\Omega^2}{240\tilde{k}^5_{\min} q_1^2 q_2} \Bigg\{ & \pi \tilde{k}^5_{\min} q_2 \Big(-15q_1^4 - 10q_1^2q_2^2 + q_2^4 \Big) \\
& + \tilde{k}^5_{\min} \left[(3q_2 - 4)(1-2q_2)^4 \mathrm{Si}(\tilde{k}_{\min} - 2\tilde{k}_{\min} q_2) + (4 - 5q_2)\mathrm{Si}(\tilde{k}_{\min}) \right] \\
& + \tilde{k}_{\min} q_1 \cos(\tilde{k}_{\min} q_1) [2 \left(\tilde{k}^2_{\min} \left(2q_1^2 + q_2^2\right) - 12 \right) \sin(\tilde{k}_{\min} q_2) \\
& + \tilde{k}_{\min} q_2 \left(\tilde{k}^2_{\min} \left(11q_1^2 + q_2^2\right) - 6 \right) \cos(\tilde{k}_{\min} q_2)] \\
& + 2\sin(\tilde{k}_{\min} q_1) \Bigg[\tilde{k}_{\min} q_2 \left(\tilde{k}^2_{\min} \left(7q_1^2 - q_2^2\right) + 6 \right) \cos(\tilde{k}_{\min} q_2) \\
& \quad + \left(\tilde{k}^4_{\min} \left(-4q_1^4 - 9q_1^2q_2^2 + q_2^4\right) + \tilde{k}^2_{\min} \left(8q_1^2 - 2q_2^2\right) + 24 \right) \sin(\tilde{k}_{\min} q_2) \Bigg] \Bigg\}
\end{aligned}
$$

$l = 1$, $m = 0$ and $q_2 < q_1$

$$
\begin{aligned}
S^{a,b}_{l,l-1}(0, R_\Omega) = \frac{R_\Omega^2}{240\tilde{k}^5_{\min} q_1^2 q_2} \Bigg\{ & 4\pi \tilde{k}^5_{\min}(q_2-1)^3 \left(6q_2^2 - 2q_2 + 1\right) \\
& + \tilde{k}^5_{\min} \left[(3q_2-4)(1-2q_2)^4 \mathrm{Si}(\tilde{k}_{\min} - 2\tilde{k}_{\min} q_2) + (4-5q_2)\mathrm{Si}(\tilde{k}_{\min})\right] \\
& + 2\tilde{k}_{\min} \cos(\tilde{k}_{\min} q_1)[4\left(-3\tilde{k}^2_{\min} q_2 + \tilde{k}^2_{\min} + 6q_2 - 6\right) \sin(\tilde{k}_{\min} q_2) \\
& + \tilde{k}_{\min}\left(11\tilde{k}^2_{\min} - 6\right) q_2 \cos(\tilde{k}_{\min} q_2)] \\
& + 2\sin(\tilde{k}_{\min} q_1)[4\left(\tilde{k}^2_{\min}\left(\tilde{k}^2_{\min}(4q_2-1) - 4q_2 + 2\right) + 6\right) \sin(\tilde{k}_{\min} q_2) \\
& + \tilde{k}_{\min}\left(7\tilde{k}^2_{\min} + 6\right) q_2 \cos(\tilde{k}_{\min} q_2)] \\
& + 2\tilde{k}^2_{\min} q_2^2 [\left(\tilde{k}^2_{\min}(2(17-6q_2)q_2 - 33) + 6\right) \cos(\tilde{k}_{\min} - 2\tilde{k}_{\min} q_2) \\
& + 2\tilde{k}_{\min}(3q_2 - 7)\sin(\tilde{k}_{\min} - 2\tilde{k}_{\min} q_2)] \Bigg\}
\end{aligned}
$$

Elliptical orbits

Coordinate change

$$t(\eta) = \Omega^{-1}\big(\eta - e\sin\eta\big)$$

$$\mathbf{r}_p(\eta) = a\big(\cos\eta - e\big)\,\hat{\mathbf{x}} + a\sqrt{1-e^2}\sin\eta\,\hat{\mathbf{y}}$$

$$\mathbf{r}_p(\eta)-\mathbf{r}_p(\eta') = \mathbf{r}_c(\eta)-\mathbf{r}_c(\eta')-\frac{ia}{\sqrt{2}}\big(1-\sqrt{1-e^2}\big)\big(\sin\eta'-\sin\eta\big)\,\big(\mathbf{e}_+ + \mathbf{e}_-\big)$$

Force

$$\mathbf{F}_{\mathrm{DF}}(t) = \big(4\pi GM\big)^2\bar{\rho}_g \int_\omega \int_{-\infty}^{+\infty} d\eta'\,\Omega^{-1}\big(1-e\cos\eta'\big)$$
$$\cdot \int_{\mathbf{k}} h\big(t(\eta')\big)\,\frac{i\mathbf{k}}{k^2}\,e^{i\mathbf{k}\cdot(\mathbf{r}_p(\eta)-\mathbf{r}_p(\eta')-i\omega(t(\eta)-t(\eta'))}$$
$$\cdot\,\tilde{G}(\mathbf{k},\omega)$$

Self-gravity

Add Self-gravity

$$\phi = \phi_0 + \phi_p$$
$$\nabla_{\mathbf{r}}\phi = 4\pi G\rho + 4\pi GM\, h(t)\, \delta^D\big(\mathbf{r}_p(t) - \mathbf{r}\big)$$

Modified scattering amplitude

$$S^{\mathrm{Sty}}_{l,l-1}(m, R_\Omega) = \lim_{\epsilon\to 0^+} \int_0^\infty d\tilde{k}\ \tilde{k}\,\frac{j_l(\tilde{k})j_{l-1}(\tilde{k})}{\tilde{k}^4/R_\Omega^2 - (m+i\epsilon)^2 + \tilde{k}_j^2}$$

Self-interaction

Modified Schrödinger equation

$$i\partial_t\psi = -\frac{\hbar}{2m_a}\Delta_{\mathbf{r}}\psi + \frac{m_a}{\hbar}\Phi\,\psi + \frac{4\pi\hbar a_s}{m_a}|\psi|^2\psi$$

Modified Scattering amplitude

$$S^{\mathrm{Sty}}_{l,l-1}(m, R_\Omega) = \lim_{\epsilon\to 0^+}\int_0^\infty d\tilde{k}\ \tilde{k}\,\frac{j_l(\tilde{k})j_{l-1}(\tilde{k})}{\tilde{k}^4/R_\Omega^2 - \tilde{\lambda}\tilde{k}^2 - (m+i\epsilon)^2}$$

Self-interaction

$m \neq 0$

$$S^{\mathrm{Sty}}_{l,l-1}(m,\ R_\Omega) = \frac{i\pi R_\Omega}{2\sqrt{\tilde{\lambda}^2 + 4m^2}}\Big[j_l(\tilde{\lambda}^{(1)})\, h^{(1)}_{l-1}(\tilde{\lambda}^{(1)}) + \frac{i}{\tilde{\lambda}^{(2)}} I_{l+1/2}\big(\tilde{\lambda}^{(2)}\big) K_{l-1/2}\big(\tilde{\lambda}^{(2)}\big)\Big]$$

$$\tilde{\lambda}^{(1)} = \sqrt{\frac{R_\Omega}{2}}\sqrt{\tilde{\lambda} + \sqrt{\tilde{\lambda}^2 + 4m^2}}$$

$$\tilde{\lambda}^{(2)} = \sqrt{\frac{R_\Omega}{2}}\sqrt{\tilde{\lambda} - \sqrt{\tilde{\lambda}^2 + 4m^2}}$$

Self-interaction

$m = 0$

$$S^{\text{Sty}}_{l,l-1}(0,\ R_\Omega) = \frac{i\pi}{2\tilde{\lambda}}\left[j_l(R_\Omega\sqrt{\tilde{\lambda}})\, h^{(1)}_{l-1}(R_\Omega\sqrt{\tilde{\lambda}}) + \frac{i}{4l^2-1}\right]$$

Orbital stagnation

Angular momentum

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times (q_1 \mathbf{F}_{\mathrm{DF},2} - q_2 \mathbf{F}_{\mathrm{DF},1})$$

$$\mathbf{L} = L\hat{\mathbf{z}}$$

$$L^2 = GM^2 \mu r_0$$

Equation for radius

$$\frac{dr_0}{dt} = -8\pi\rho \sqrt{\frac{Gr_0^5}{\mu}} \left(q_1 \Im(I_2) + q_2 \Im(I_1)\right)$$

$$\frac{dr_0}{dt} = -8\pi\rho \sqrt{\frac{Gr_\Omega^5}{\mu}} \left[\frac{8}{5\pi} a_{\mathrm{GW}} \left(\frac{r_0}{r_\Omega}\right)^{-3} + \left(\frac{r_0}{r_\Omega}\right)^{5/2} \left(q_1 \Im(I_2) + q_2 \Im(I_1)\right)\right]$$