Computing the Trispectrum for Axion Inflation

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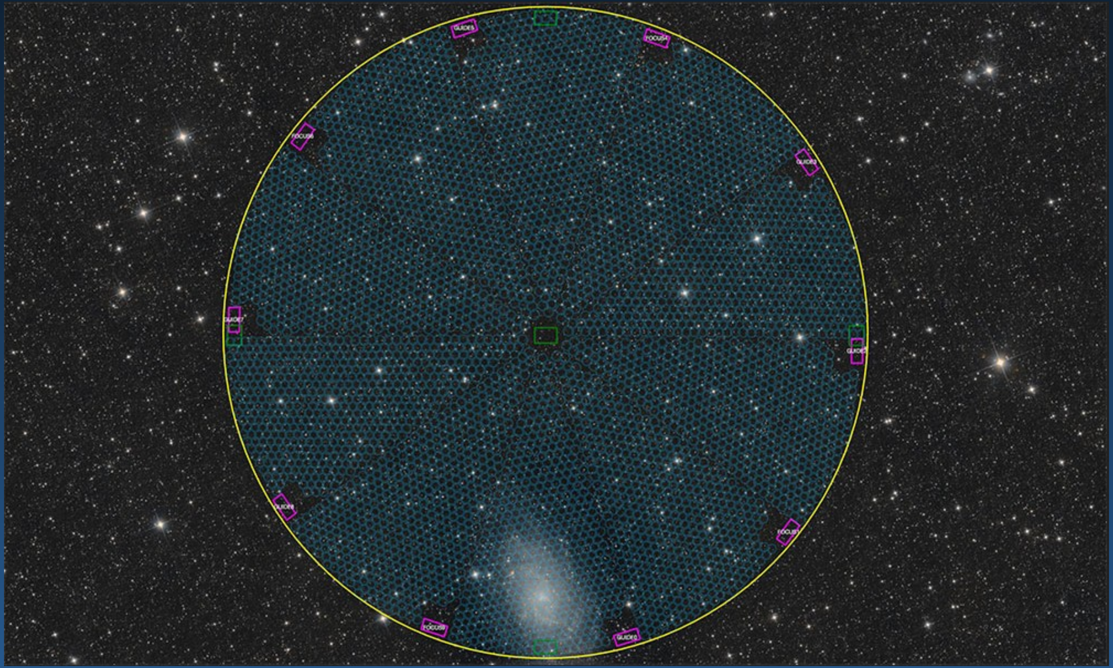


Cosmology from Home 2023

Symmetries of Nature

- The laws of physics are invariant under:

  - Time translations

  - Spatial translations

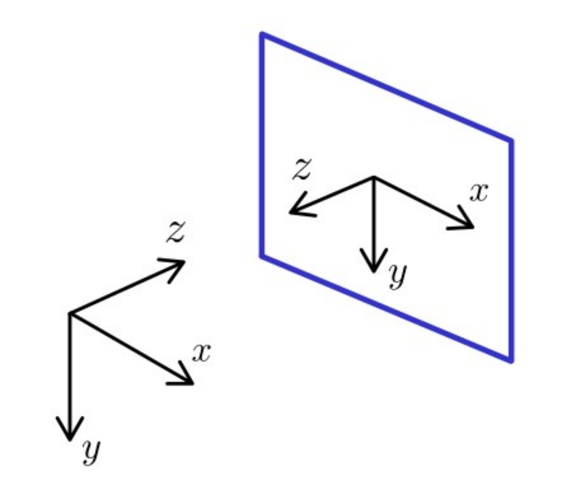
  - Rotations

- All known fundamentals forces are parity-symmetric except for the weak force.

Parity

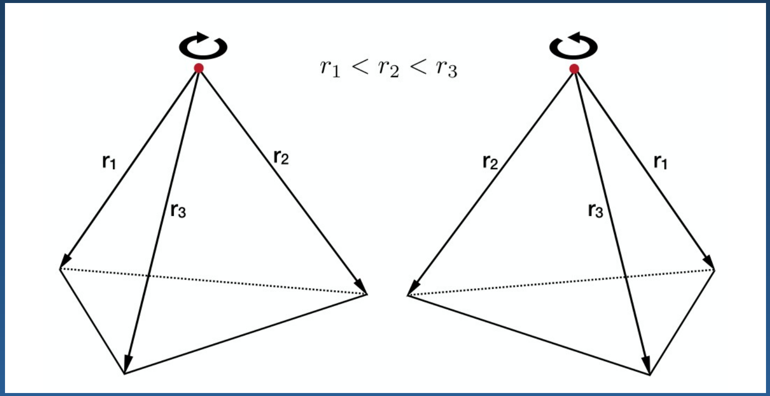
- A parity transformation is an inversion of all spatial coordinates.
- Parity transformations make a right hand appear as a left hand in the mirror.

$$P : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$



Large scale structure

- The universe is highly homogeneous and isotropic, but what about handedness?
- Are there more right handed or left handed configurations of galaxies?



Evidence for Parity Violation

- Hou, Slepian, and Cahn 2022 reported evidence of parity violation at 7.1σ, also see Philcox 2022.
- The 4-Point Correlation Function (4PCF) is the simplest statistic sensitive to parity violation.

$$\begin{aligned}\hat{\zeta}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) &\equiv \langle \delta(\mathbf{s})\delta(\mathbf{s}+\mathbf{r}_1)\delta(\mathbf{s}+\mathbf{r}_2)\delta(\mathbf{s}+\mathbf{r}_3)\rangle \\ &= \int \frac{d\mathbf{s}}{V}\,\delta(\mathbf{s})\delta(\mathbf{s}+\mathbf{r}_1)\delta(\mathbf{s}+\mathbf{r}_2)\delta(\mathbf{s}+\mathbf{r}_3),\end{aligned}$$

$$\delta(\mathbf{s}) \equiv \rho(\mathbf{s})/\bar{\rho} - 1$$

Hou et al. 2022 arXiv:2206.03625 Philcox 2022 arXiv:2206.03625

What Caused Parity Violation?

- All fundamental interactions are parity-symmetric except for the weak interaction.
- Any parity violation in the LSS must have come from new physics early in the universe.
- Parity-violating interactions during inflation could seed the parity violation that we see today.

Cahn et al. 2021 arXiv:2110.12004

Motivation for axions

- QCD Lagrangian contains a CP violating term

$$L_\theta = \theta \frac{g^2}{32\pi^2} F_a^{\mu\nu} \tilde{F}_{a\mu\nu}.$$

- Experimental bounds give $\theta < 10^{-9}$.
- Strong CP Problem: Why is θ so small (possibly zero)?
- Problem solved by introduction of a global U(1) Peccei-Quinn (PQ) symmetry. The breaking of this symmetry leads to production of axions.
- Vacuum expectation value of the axion field exactly cancels the CP-violating term in the Lagrangian.

Peccei 2006 arXiv:hep-ph/0607268

Axion Inflation

- Standard model of inflation preserves parity.
- Axions are pseudo-scalars; they pick up a minus sign under parity.
- We consider an inflationary model with an axion coupled to a U(1) gauge field.

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R - \frac{1}{2}(\partial\varphi)^2 - V(\varphi) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\alpha}{4f}\varphi\,\tilde{F}^{\mu\nu}F_{\mu\nu}\right]$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathbf{A}(\tau,\mathbf{x}) = \sum_{\lambda=\pm}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\epsilon_\lambda(\mathbf{k})a_\lambda(\mathbf{k})A_\lambda(\tau,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \mathrm{h.c.}\right]$$

$$A_+(\tau,k) \cong \frac{1}{\sqrt{2k}}\left(\frac{k}{2\xi aH}\right)^{1/4} e^{\pi\xi - 2\sqrt{2\xi k/(aH)}}$$

Barnaby et al. arXiv:1011.1500

Computing the Trispectrum

- The density curvature perturbations are defined as $\zeta = -H\delta\phi/\dot{\phi}$.
- Inflaton perturbations

$$\delta\phi(\tau,\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{Q(\tau,\mathbf{k})}{a(\tau)}e^{-i\mathbf{k}\cdot\mathbf{x}}$$

- The inflaton perturbations can be separated into vacuum modes and an inverse decay modes.

$$Q(\tau,\mathbf{k}) = Q^{\mathrm{vac}}(\tau,\mathbf{k}) + Q^{\mathrm{inv}}(\tau,\mathbf{k})$$

- The density curvature perturbation trispectrum is given by

$$\langle\zeta(\tau,\mathbf{k}_1)\zeta(\tau,\mathbf{k}_2)\zeta(\tau,\mathbf{k}_3)\zeta(\tau,\mathbf{k}_4)\rangle = \frac{H^4}{\dot{\phi}\,a^4}\left[\langle Q^{\mathrm{vac}}(\tau,\mathbf{k}_1)Q^{\mathrm{vac}}(\tau,\mathbf{k}_2)Q^{\mathrm{vac}}(\tau,\mathbf{k}_3)Q^{\mathrm{vac}}(\tau,\mathbf{k}_4)\rangle + \langle Q^{\mathrm{inv}}(\tau,\mathbf{k}_1)Q^{\mathrm{inv}}(\tau,\mathbf{k}_2)Q^{\mathrm{inv}}(\tau,\mathbf{k}_3)Q^{\mathrm{inv}}(\tau,\mathbf{k}_4)\rangle\right]$$

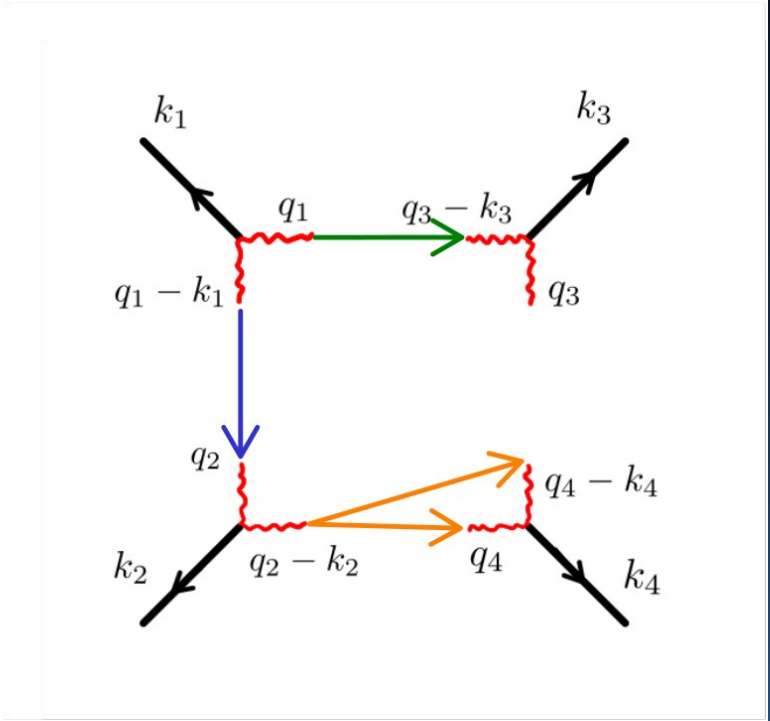
Source Function

$$Q^{\rm inv}(\tau,\mathbf{k}) = \int_{-\infty}^{0} \mathrm{d}\tau' G(\tau,\tau',k) J(\tau',\mathbf{k})$$

$$J(\tau,\mathbf{k}) = -\frac{1}{\Lambda a(\tau)} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, q[\epsilon_+(\mathbf{k}-\mathbf{q})\cdot\epsilon_+(\mathbf{q})]$$
$$\times \Big\{ a_+(\mathbf{k}-\mathbf{q}) A'_+(\tau,|\mathbf{k}-\mathbf{q}|) a_+(\mathbf{q}) A_+(\tau,q) + a_+(\mathbf{k}-\mathbf{q}) A'_+(\tau,|\mathbf{k}-\mathbf{q}|) a^\dagger_+(-\mathbf{q}) A^*_+(\tau,q)$$
$$+ a^\dagger_+(\mathbf{q}-\mathbf{k}) A'^*_+(\tau,|\mathbf{q}-\mathbf{k}|) a_+(\mathbf{q}) A_+(\tau,q) + a^\dagger_+(\mathbf{q}-\mathbf{k}) A'^*_+(\tau,|\mathbf{q}-\mathbf{k}|) a^\dagger_+(-\mathbf{q}) A^*_+(\tau,q) \Big\}.$$

$$\langle \zeta(\tau,\mathbf{k}_1)\zeta(\tau,\mathbf{k}_2)\zeta(\tau,\mathbf{k}_3)\zeta(\tau,\mathbf{k}_4)\rangle$$
$$\propto \int_{-\infty}^{0}\mathrm{d}\tau_1 \int_{-\infty}^{0}\mathrm{d}\tau_2 \int_{-\infty}^{0}\mathrm{d}\tau_3 \int_{-\infty}^{0}\mathrm{d}\tau_4 G(\tau,\tau_1,\mathbf{k}_1) G(\tau,\tau_2,\mathbf{k}_2) G(\tau,\tau_3,\mathbf{k}_2) G(\tau,\tau_4,\mathbf{k}_2)$$
$$\times \langle J(\tau_1,\mathbf{k}_2) J(\tau_2,\mathbf{k}_2) J(\tau_3,\mathbf{k}_3) J(\tau_4,\mathbf{k}_4)\rangle\,.$$

Counting Contractions



Evaluation of the trispectrum involves integrating
6 x 4 x 2 = 48 different terms!

A single contraction

$$\langle J(\tau,\mathbf{k_1})J(\tau,\mathbf{k_2})J(\tau,\mathbf{k_3})J(\tau,\mathbf{k_4})\rangle^{(1)} =$$

$$\int\int\int\int \mathrm{d}^3\mathbf{q_1}\ \mathrm{d}^3\mathbf{q_2}\ \mathrm{d}^3\mathbf{q_3}\ \mathrm{d}^3\mathbf{q_4}\ [\epsilon(\mathbf{q_1})\cdot\epsilon(\mathbf{k_1}-\mathbf{q_1})]\ [\epsilon(\mathbf{q_2})\cdot\epsilon(\mathbf{k_2}-\mathbf{q_2})]\ [\epsilon(\mathbf{q_3})\cdot\epsilon(\mathbf{k_3}-\mathbf{q_3})]$$
$$\times [\epsilon(\mathbf{q_4})\cdot\epsilon(\mathbf{k_4}-\mathbf{q_4})]\ \delta_{\mathrm{D}}^{[3]}(\mathbf{q_2}+\mathbf{k_4}-\mathbf{q_4})\delta_{\mathrm{D}}^{[3]}(\mathbf{q_1}+\mathbf{k_3}-\mathbf{q_3})\delta_{\mathrm{D}}^{[3]}(\mathbf{k_1}-\mathbf{q_1}+\mathbf{q_4})$$
$$\times \delta_{\mathrm{D}}^{[3]}(\mathbf{k_2}-\mathbf{q_2}+\mathbf{q_3})A'_+(\tau_1,|\mathbf{k_1}-\mathbf{q_1}|)A_+(\tau_1,q_1)A'_+(\tau_2,|\mathbf{k_2}-\mathbf{q_2}|)A_+(\tau_2,q_2)$$
$$\times A'^*_+(\tau_3,|\mathbf{k_3}-\mathbf{q_3}|)A^*_+(\tau_3,q_3)A'^*_+(\tau_4,|\mathbf{k_4}-\mathbf{q_4}|)A^*_+(\tau_4,q_4)$$

- If we were to integrate out all delta functions we would be left with a 12-dimensional integral.
- We will reduce the dimensionality by expanding the delta functions.

Dirac Delta Expansions

$$\delta_{\rm D}^{[3]}(\mathbf{q}+\mathbf{p}) = (4\pi)^2 \sum_{\ell=0}^{\infty} (-1)^\ell \mathcal{R}_{\ell\ell}(q,p) \sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\hat{\mathbf{q}}) Y_\ell^m(\hat{\mathbf{p}}).$$

$$\delta_{\rm D}^{[3]}(\mathbf{k}+\mathbf{q}+\mathbf{p}) = (4\pi)^3 \sum_{\ell_1\ell_2\ell_3} \sum_{m_1m_2m_3} i^{\ell_1+\ell_2+\ell_3} \mathcal{G}_{\ell_1\ell_2\ell_3}^{m_1m_2m_3} \mathcal{R}_{\ell_1\ell_2\ell_3}(k,q,p) Y_{\ell_1}^{m_1*}(\hat{\mathbf{k}}) Y_{\ell_2}^{m_2*}(\hat{\mathbf{q}}) Y_{\ell_3}^{m_3*}(\hat{\mathbf{p}})$$

$$\delta_{\rm D}^{[3]}(\mathbf{k}+\mathbf{q}+\mathbf{p}+\mathbf{s}) = (4\pi)^4 \sum_{\ell_1\ell_2\ell_3\ell_4} \sum_{m_1m_2m_3m_4} \sum_{L,M} i^{\ell_1+\ell_2+\ell_3+\ell_4} \mathcal{K}_{\ell_1\ell_2\ell_3\ell_4L}^{m_1m_2m_3m_4M} \mathcal{R}_{\ell_1\ell_2\ell_3\ell_4}(k,q,p,s)$$
$$\times\, Y_{\ell_1}^{m_1*}(\hat{\mathbf{k}}) Y_{\ell_2}^{m_2*}(\hat{\mathbf{q}}) Y_{\ell_3}^{m_3*}(\hat{\mathbf{p}}) Y_{\ell_4}^{m_4*}(\hat{\mathbf{s}})$$

$$\mathcal{R}_{\ell\ell}(p,q) \equiv \int {\rm d}x\; x^2 j_\ell(px) j_\ell(qx)$$

$$\mathcal{R}_{\ell_1\ell_2\ell_3}(k,p,q) \equiv \int {\rm d}x\; x^2 j_{\ell_1}(kx) j_{\ell_2}(px) j_{\ell_3}(qx)$$

$$\mathcal{R}_{\ell_1\ell_2\ell_3\ell_4}(k,p,q,s) \equiv \int {\rm d}x\; x^2 j_{\ell_1}(kx) j_{\ell_2}(px) j_{\ell_3}(qx) j_{\ell_4}(qs)$$

The Contraction Rewritten

$$\langle J(\tau,\mathbf{k_1})J(\tau,\mathbf{k_2})J(\tau,\mathbf{k_3})J(\tau,\mathbf{k_4})\rangle^{(1)} = \tag{5.2}$$

$$\sum_{\boldsymbol{\ell},\mathbf{m}}\sum_{\mathbf{a},\mathbf{b}}\sum_{\mathbf{c},\mathbf{d}}\sum_{\mathbf{e},\mathbf{f}}(4\pi)^{12} i^{\sum_i(\ell_i+a_i+c_i+d_i)} Y_{\ell_2}^{m_2*}(\hat{\mathbf{k}}_4)Y_{c_1}^{d_1*}(\hat{\mathbf{k}}_1)Y_{a_2}^{b_2*}(\hat{\mathbf{k}}_3)Y_{e_1}^{f_1*}(\hat{\mathbf{k}}_2)$$
$$\times\, \mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}\mathcal{G}^{d_1d_2d_3}_{c_1c_2c_3}\mathcal{G}^{b_1b_2b_3}_{a_1a_2a_3}\mathcal{G}^{f_1f_2f_3}_{e_1e_2e_3}$$
$$\times \int\!\!\int\!\!\int\!\!\int \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\, x_1^2x_2^2x_3^2x_4^2 j_{e_1}(k_2x_4)j_{\ell_2}(k_4x_1)j_{a_2}(k_3x_2)j_{c_1}(k_1x_3)$$
$$\times \int \mathrm{d}^3\mathbf{q_4}\; q_4[\epsilon(\mathbf{q_4})\cdot\epsilon(\mathbf{k_4}-\mathbf{q_4})]Y_{\ell_3}^{m_3*}(-\hat{\mathbf{q}}_4)Y_{c_3}^{d_3*}(\hat{\mathbf{q}}_4)A_+'^{*}(\tau_4,|\mathbf{k_4}-\mathbf{q_4}|)A_+^*(\tau_4,q_4)j_{\ell_3}(q_4x_1)j_{c_3}(q_4x_3)$$
$$\int \mathrm{d}^3\mathbf{q_3}\; q_3[\epsilon(\mathbf{q_3})\cdot\epsilon(\mathbf{k_3}-\mathbf{q_3})]Y_{a_3}^{b_3*}(-\hat{\mathbf{q}}_3)Y_{e_3}^{f_3*}(\hat{\mathbf{q}}_3)A_+'^{*}(\tau_3,|\mathbf{k_3}-\mathbf{q_3}|)A_+^*(\tau_3,q_3)j_{e_3}(q_3x_4)j_{a_3}(q_3x_2)$$
$$\int \mathrm{d}^3\mathbf{q_2}\; q_2[\epsilon(\mathbf{q_2})\cdot\epsilon(\mathbf{k_2}-\mathbf{q_2})]Y_{e_2}^{f_2*}(-\hat{\mathbf{q}}_2)Y_{\ell_1}^{m_1*}(\hat{\mathbf{q}}_2)A_+'(\tau_2,|\mathbf{k_2}-\mathbf{q_2}|)A_+(\tau_2,q_2)j_{e_2}(q_2x_4)j_{\ell_1}(q_2x_1)$$
$$\int \mathrm{d}^3\mathbf{q_1}\; q_1[\epsilon(\mathbf{q_1})\cdot\epsilon(\mathbf{k_1}-\mathbf{q_1})]Y_{c_2}^{d_2*}(-\hat{\mathbf{q}}_1)Y_{a_1}^{b_1*}(\hat{\mathbf{q}}_1)A_+'(\tau_1,|\mathbf{k_1}-\mathbf{q_1}|)A_+(\tau_1,q_1)j_{a_1}(q_1x_2)j_{c_2}(q_1x_3)$$

Recognizing the Convolution

$$\mathrm{I}^{mm'(1)}_{\ell\ell'}(\mathbf{k},x,x',\tau) = \int \mathrm{d}^3\mathbf{q}\; q[\epsilon(\mathbf{q})\cdot\epsilon(\mathbf{k}-\mathbf{q})]Y^{m*}_{\ell}(-\hat{\mathbf{q}})Y^{m'*}_{\ell'}(\hat{\mathbf{q}})A'_{+}(\tau,|\mathbf{k}-\mathbf{q}|)A_{+}(\tau,q) \times j_{\ell}(qx)j_{\ell'}(qx').$$

$$\epsilon(\mathbf{q})\cdot\epsilon(\mathbf{k}-\mathbf{q}) = \frac{1}{2}\Bigg[\cos\alpha\cos\theta\cos\phi\cos\beta + \cos\alpha\cos\theta\sin\phi\sin\beta - \cos\phi\cos\beta - \sin\phi\sin\beta + \sin\alpha\sin\theta + i\left(\hat{\mathbf{q}}\times\frac{\mathbf{k}-\mathbf{q}}{|\mathbf{k}-\mathbf{q}|}\cdot\mathbf{z}\right)\frac{\cos\alpha-\cos\theta}{\sin\alpha\sin\theta}\Bigg]$$

Convolution:

$$[f\star g](\mathbf{k}) = \int \mathrm{d}^3\mathbf{q}\; f(\mathbf{q})g(\mathbf{k}-\mathbf{q})$$

Convolution Theorem:

$$[f\star g](\mathbf{k}) = \mathcal{F}\left\{\mathcal{F}^{-1}\{f(\mathbf{q})\}\mathcal{F}^{-1}\{g(\mathbf{q})\}\right\}(\mathbf{k})$$

Final integrals

$$\mathrm{I}_{\ell\ell'}^{mm'(1)}(\mathbf{k}, x, x', \tau) \propto k^{\delta+\gamma-1} \int_0^\infty \mathrm{d}u\, j_l(kux) j_{l'}(kux') u^\delta C_\gamma(u)$$

- We have been able to analytically compute all angular integrals.
- We are left with only radial integrals, greatly reducing the dimensionality of integration.
- Numerical calculation of the trispectrum for axion inflation is now much less computationally intensive.

Conclusion

- We have shown how to simplify to calculation of the trispectrum for axion-gauge field inflation.
- We will use these techniques to numerically compute the trispectrum for a large range of external momenta combinations.
- We will use the DESI data to make a robust measurement of the Galaxy 4PCF and search for signatures of this axion-gauge field inflationary model and constrain the physics of the early universe.

References

- “Measurement of Parity-Odd Modes in the Large-Scale 4-Point Correlation Function of SDSS BOSS DR12 CMASS and LOWZ Galaxies” arXiv:2206.03625
- “Large Nongaussianity in Axion Inflation” arXiv:1011.1500
- “Phenomenology of a Pseudo-Scalar Inflaton: Naturally Large Nongaussianity” arXiv:1102.4333
- “Probing Parity-Violation with the Four-Point Correlation Function of BOSS Galaxies” arXiv:2206.04227
- “A Test for Cosmological Parity Violation Using the 3D Distribution of Galaxies” arXiv:2110.12004
- “The Strong CP Problem and Axions” arXiv:hep-ph/0607268

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