Analytic Covariance Matrices for N-Point Correlation Functions

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Cosmology from Home
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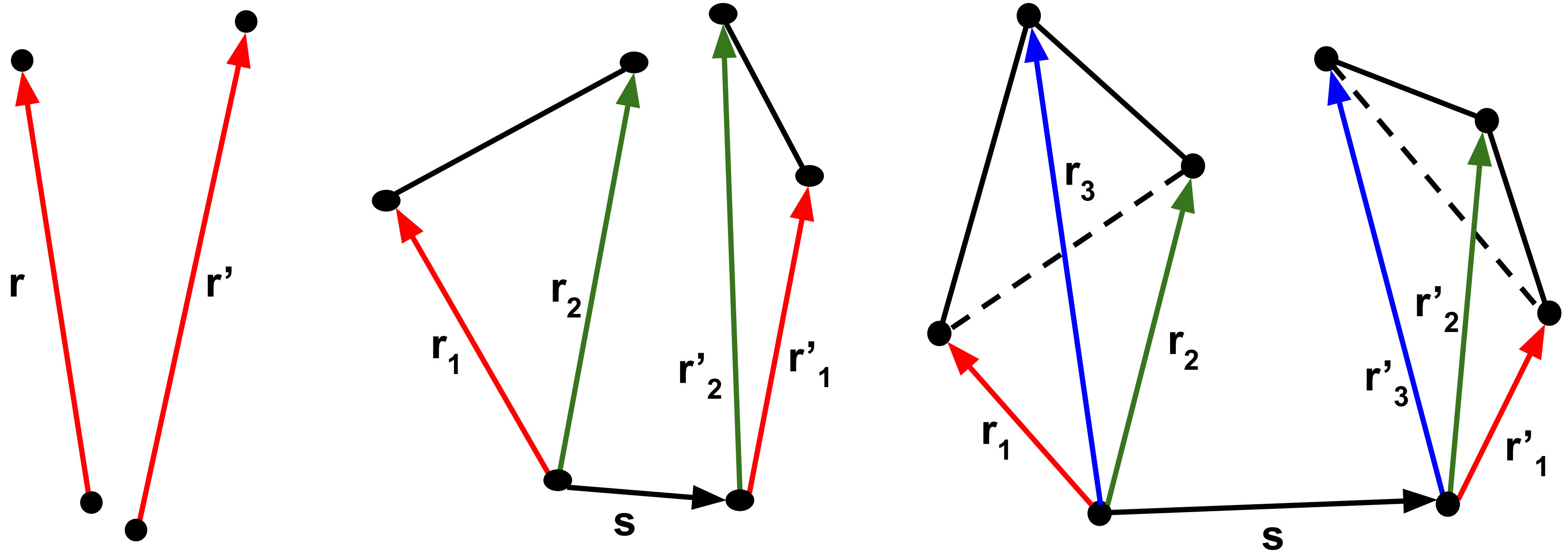
N-Point Correlation Functions (NPCFs)



Background and Motivation

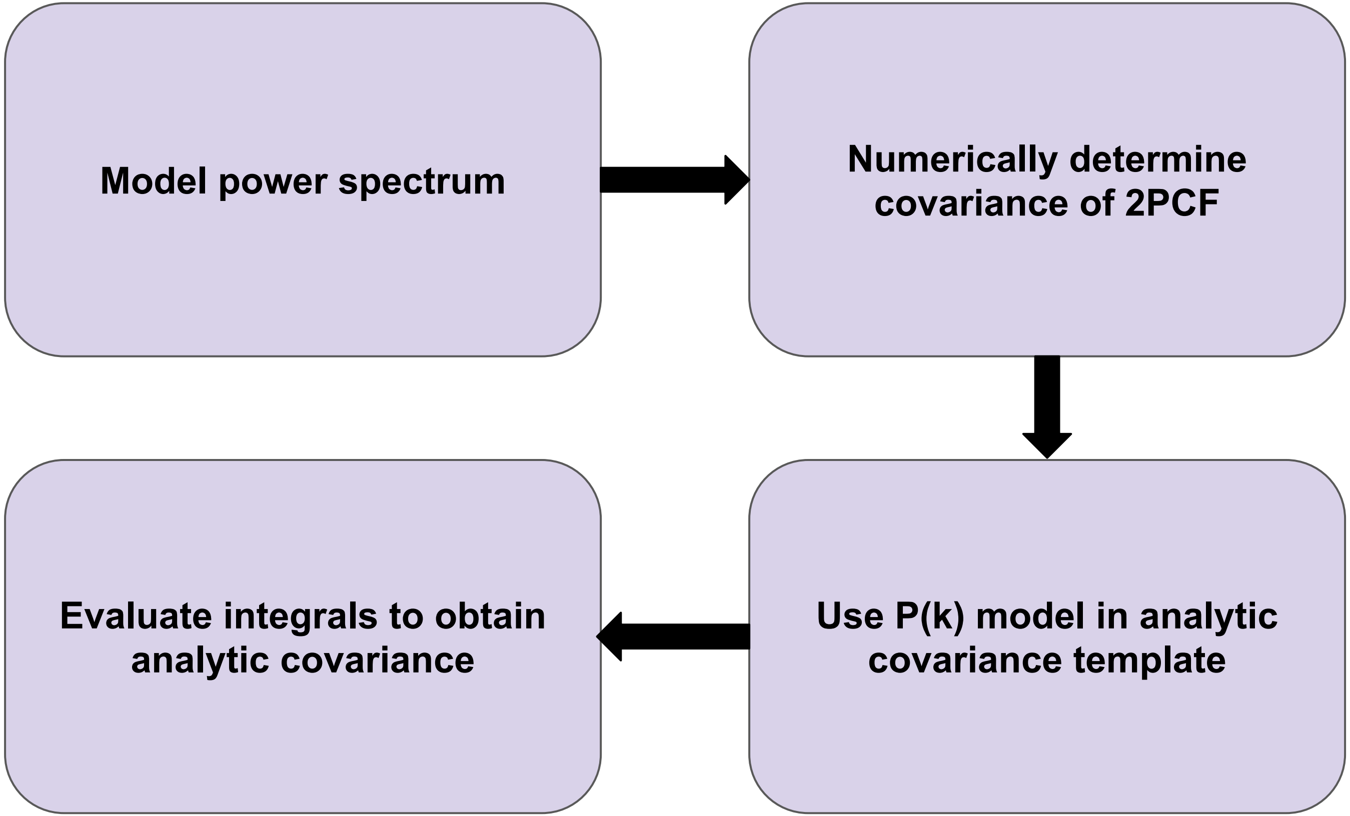
Covariance Matrix

- Obtained from mock catalogs
- Challenge: high dimensionality of observables
- Need several mocks per degree of freedom for smoothly invertible matrix
- Gaussian random field (GRF) limit
- Analytic covariance based on leading order result

Power Spectrum P(k)

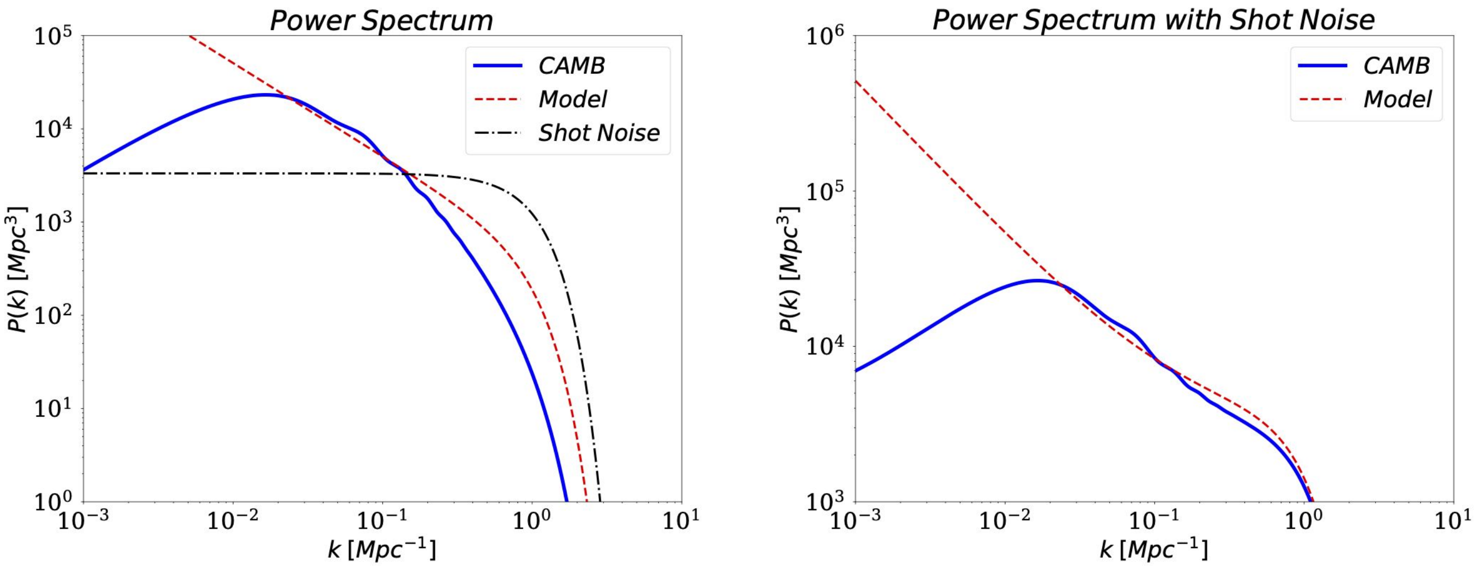
- Traces clustering of matter as a function of scale
- Fourier transform of 2PCF
- Analytic covariance requires integrals of P(k)
- Analytic solution to integrals allows for better understanding of the structure of the covariance

Outline for Determining Analytic Covariance Matrices



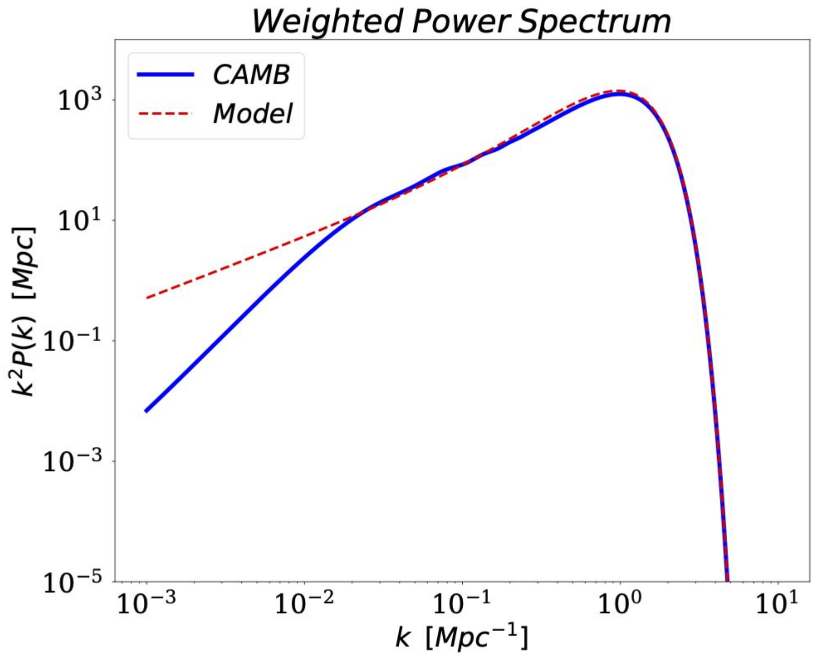
A Model for the Power Spectrum

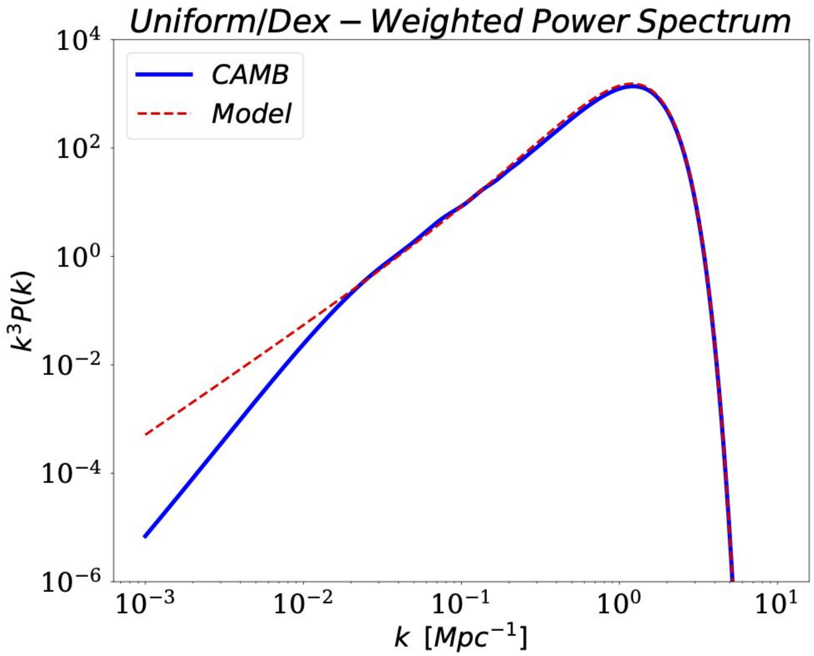
- Model: $P(k) = \frac{A}{k} + \frac{1}{\bar{n}}$
- $A = 509\ \mathrm{Mpc}^2,\ \bar{n} = 3 \times 10^{-4}\ \mathrm{Mpc}^{-3}$
- True P(k) from CAMB
- *Planck* 2018: $H_0 = 67.36\ \mathrm{km\ s^{-1}\ Mpc^{-1}},\ \Omega_b h^2 = 0.02237,\ \Omega_c h^2 = 0.1200,\ n_s = 0.9649$



A Model for the Power Spectrum

- Covariance requires integrating P(k) against $k^2 \, dk$
- $\frac{d}{dk}(\ln k) = \frac{1}{k} \rightarrow k^2 \, dk = k^3 \, d(\ln k)$





Half-Inverse Test

- Data weighted by inverse covariance
- Half-inverse test: $$S \equiv C_{\text{model}}^{-1/2} C_{\text{true}} C_{\text{model}}^{-1/2} - \mathbb{1}$$

- Numerically determine 2PCF covariance from model and true power spectra:

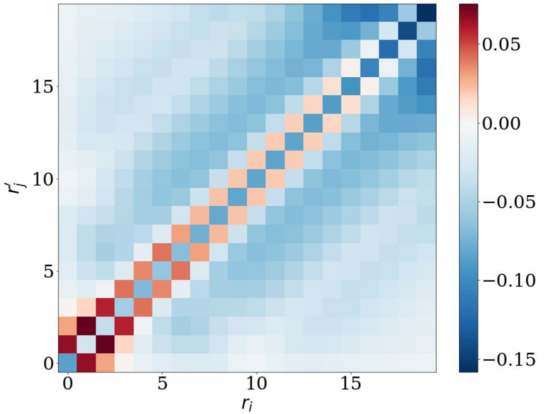
$$C_{ij} = \frac{2}{V} \int_0^\infty \frac{k^2 dk}{2\pi^2} \Delta j_1(kr_i) \Delta j_1(kr'_j) P^2(k)$$

$$\Delta j_1(kr_i) = \left(\frac{3}{r_{i2}^3 - r_{i1}^3} \right) \frac{r_{i2}^2 j_1(kr_{i2}) - r_{i1}^2 j_1(kr_{i1})}{k}$$

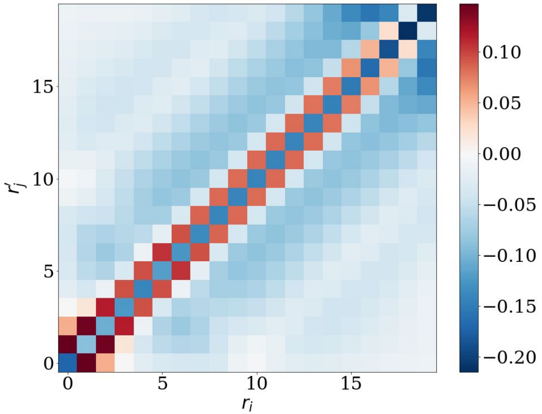
Xu et al. 2012

- Separations from 0 Mpc to 200 Mpc, bin width = 10 Mpc, V = 2 cubic Gpc

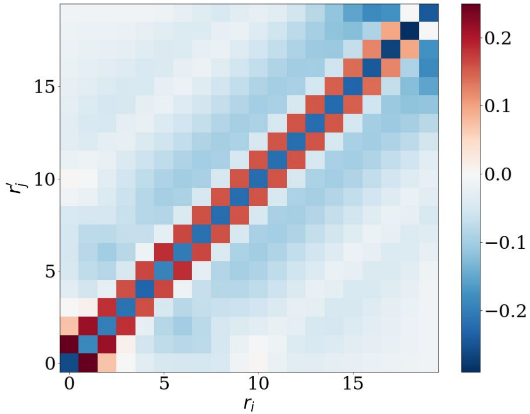
Half-Inverse Test



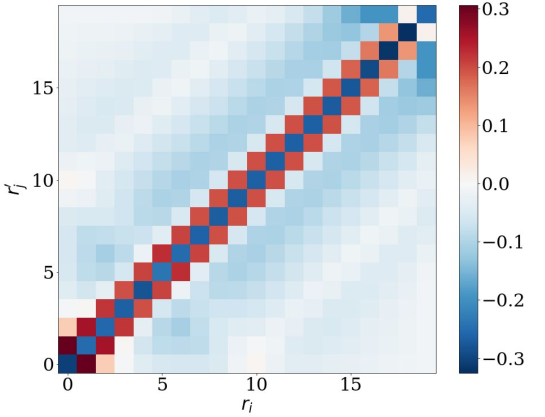
(a) $\bar{n} = 10^{-4}$ Mpc^{-3}, RMS = 3.22%



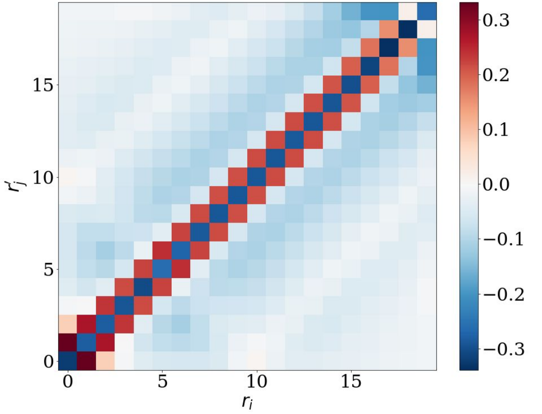
(b) $\bar{n} = 3 \times 10^{-4}$ Mpc^{-3}, RMS = 5.43%



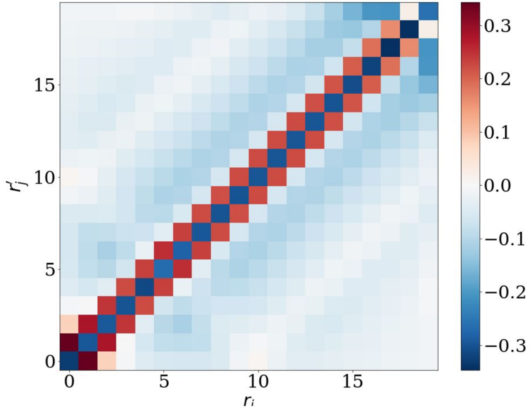
(c) $\bar{n} = 10^{-3}$ Mpc^{-3}, RMS = 8.19%



(d) $\bar{n} = 3 \times 10^{-3}$ Mpc^{-3}, RMS = 9.76%



(e) $\bar{n} = 10^{-2}$ Mpc^{-3}, RMS = 10.5%



(f) $\bar{n} = \infty$ Mpc^{-3}, RMS = 10.8%

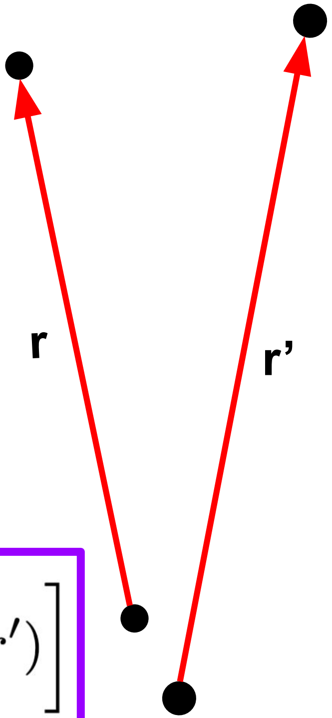
Analytic Covariance Matrix for 2PCF

$$\mathrm{Cov}(r, r') = \frac{2}{V}\int_0^\infty \frac{k^2 dk}{2\pi^2} j_0(kr) j_0(kr') P^2(k)$$ Xu et al. 2012

$$P(k) = \frac{A}{k} + \frac{1}{\bar{n}}$$

$$\mathrm{Cov}(r, r') = \frac{1}{\pi^2 V}\int_0^\infty k^2 dk\, j_0(kr) j_0(kr') \left[\left(\frac{A}{k}\right)^2 + \frac{2A}{\bar{n}k} + \frac{1}{\bar{n}^2}\right]$$

$$\mathrm{Cov}(r, r') = \frac{1}{\pi^2 V r r'}\left[\frac{\pi A^2}{2} r + \frac{2A}{\bar{n}} \tanh^{-1}\left(\frac{r}{r'}\right) + \frac{\pi}{2\bar{n}^2}\delta_{\mathrm{D}}^{[1]}(r - r')\right]$$



Analytic Covariance Matrix for 3PCF

- Legendre basis (Slepian and Eisenstein 2015)
- Isotropic basis:

$$\mathrm{Cov}_{\ell,\ell'}(r_1,r_2;r_1',r_2') = (4\pi)^3 \sum_{\ell''} \sqrt{(2\ell+1)(2\ell'+1)}(2\ell''+1) \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix}^2$$

$$\times \int_0^\infty \frac{s^2 ds}{V} \Big\{ (-1)^{\ell''} \Big[\xi(s) f_{\ell,\ell',\ell''}(r_1,r_1',s) f_{\ell,\ell',\ell''}(r_2,r_2',s)$$

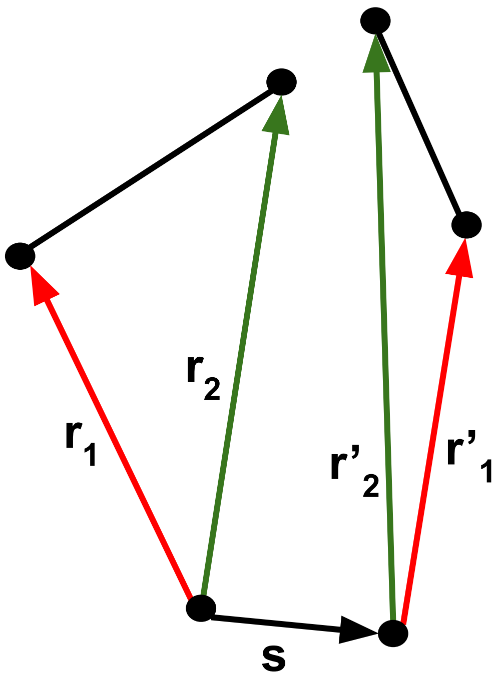
$$+ \xi(s) f_{\ell,\ell',\ell''}(r_1,r_2',s) f_{\ell,\ell',\ell''}(r_2,r_1',s) \Big]$$

$$+(-1)^{(\ell+\ell'+\ell'')/2} \Big[f_{0,\ell',\ell'}(0,r_1',s) f_{\ell,0,\ell}(r_1,0,s) f_{\ell,\ell',\ell''}(r_2,r_2',s)$$

$$+ f_{0,\ell',\ell'}(0,r_1',s) f_{\ell,\ell',\ell''}(r_1,r_2',s) f_{\ell,0,\ell}(r_2,0,s)$$

$$+ f_{0,\ell',\ell'}(0,r_2',s) f_{\ell,0,\ell}(r_1,0,s) f_{\ell,\ell',\ell''}(r_2,r_1',s)$$

$$+ f_{0,\ell',\ell'}(0,r_2',s) f_{\ell,\ell',\ell''}(r_1,r_1',s) f_{\ell,0,\ell}(r_2,0,s) \Big] \Big\}.$$



2PCF From P(k) Model

- 2PCF is the inverse Fourier transform of the power spectrum

$$\xi(s) = \int_0^\infty \frac{k^2 dk}{2\pi^2} P(k) j_0(ks)$$

$$P(k) = \frac{A}{k} + \frac{1}{\bar{n}}$$

$$\xi(s) = \frac{A}{2\pi^2 s^2} + \frac{1}{4\pi \bar{n} s^2} \delta_{\mathrm{D}}^{[1]}(s)$$

f-Integrals

$$f_{\ell,\ell',\ell''}(r_i, r'_j, s) \equiv \int_0^\infty \frac{k^2 dk}{2\pi^2} \; P(k) j_\ell(kr_i) j_{\ell'}(kr'_j) j_{\ell''}(ks)$$

Hou et al. 2021

$$f_{\ell,\ell',\ell''}(r_i, r'_j, s) = \int_0^\infty \frac{k^2 dk}{2\pi^2} \; \left(\frac{A}{k} + \frac{1}{\bar{n}}\right) j_\ell(kr_i) j_{\ell'}(kr'_j) j_{\ell''}(ks)$$

- Two cases needed for covariance of 3PCF: $f_{0,\ell,\ell}(0, r_j, s)$ and $f_{\ell,\ell',\ell''}(r_i, r_j, s)$
- First case: $f_{0,\ell,\ell}(0, r_j, s) = \int_0^\infty \frac{k^2 dk}{2\pi^2} \; \left(\frac{A}{k} + \frac{1}{\bar{n}}\right) j_\ell(kr_j) j_\ell(ks)$

$$f_{0,\ell,\ell}(0, r_j, s) = \frac{1}{4\pi^2} \left[\frac{A\sqrt{\pi}(r_j s)^\ell \, \Gamma(\ell+1)}{(r_j+s)^{2(\ell+1)} \, \Gamma(\ell+3/2)} \, {}_2F_1\left(\ell+1, \ell+1; 2(\ell+1); \frac{4 r_j s}{(r_j+s)^2}\right) + \frac{\pi}{\bar{n} r_j s} \delta_{\rm D}^{[1]}(r_j - s) \right]$$

f-Integrals

- Second case: $f_{\ell,\ell',\ell''}(r_i, r'_j, s) \equiv \int_0^\infty \frac{k^2 dk}{2\pi^2} P(k) j_\ell(kr_i) j_{\ell'}(kr'_j) j_{\ell''}(ks)$

$$f_{\ell,\ell',\ell''}(r_i, r'_j, s) = \int_0^\infty \frac{k^2 dk}{2\pi^2} \left(\frac{A}{k} + \frac{1}{\bar{n}}\right) j_\ell(kr_i) j_{\ell'}(kr'_j) j_{\ell''}(ks)$$

- Define two new integrals:

$$I^{[3,\mathrm{lin}]}_{\ell,\ell',\ell''}(r_i, r'_j, s) \equiv \int_0^\infty k dk \; j_\ell(kr_i) j_{\ell'}(kr'_j) j_{\ell''}(ks)$$

$$I^{[3,\mathrm{quad}]}_{\ell,\ell',\ell''}(r_i, r'_j, s) \equiv \int_0^\infty k^2 dk \; j_\ell(kr_i) j_{\ell'}(kr'_j) j_{\ell''}(ks)$$

- Rewrite f integral as

$$f_{\ell,\ell',\ell''}(r_i, r'_j, s) = \frac{1}{2\pi^2} \left[A I^{[3,\mathrm{lin}]}_{\ell,\ell',\ell''}(r_i, r'_j, s) + \frac{1}{\bar{n}} I^{[3,\mathrm{quad}]}_{\ell,\ell',\ell''}(r_i, r'_j, s) \right]$$

f-Integrals

$$I^{[3,\mathrm{quad}]}_{\ell,\ell',\ell''}(r_i, r'_j, s) = \frac{\pi}{2} W_{\ell,\ell',\ell''}(r_i, r'_j) s^{2b-\ell''-1}$$

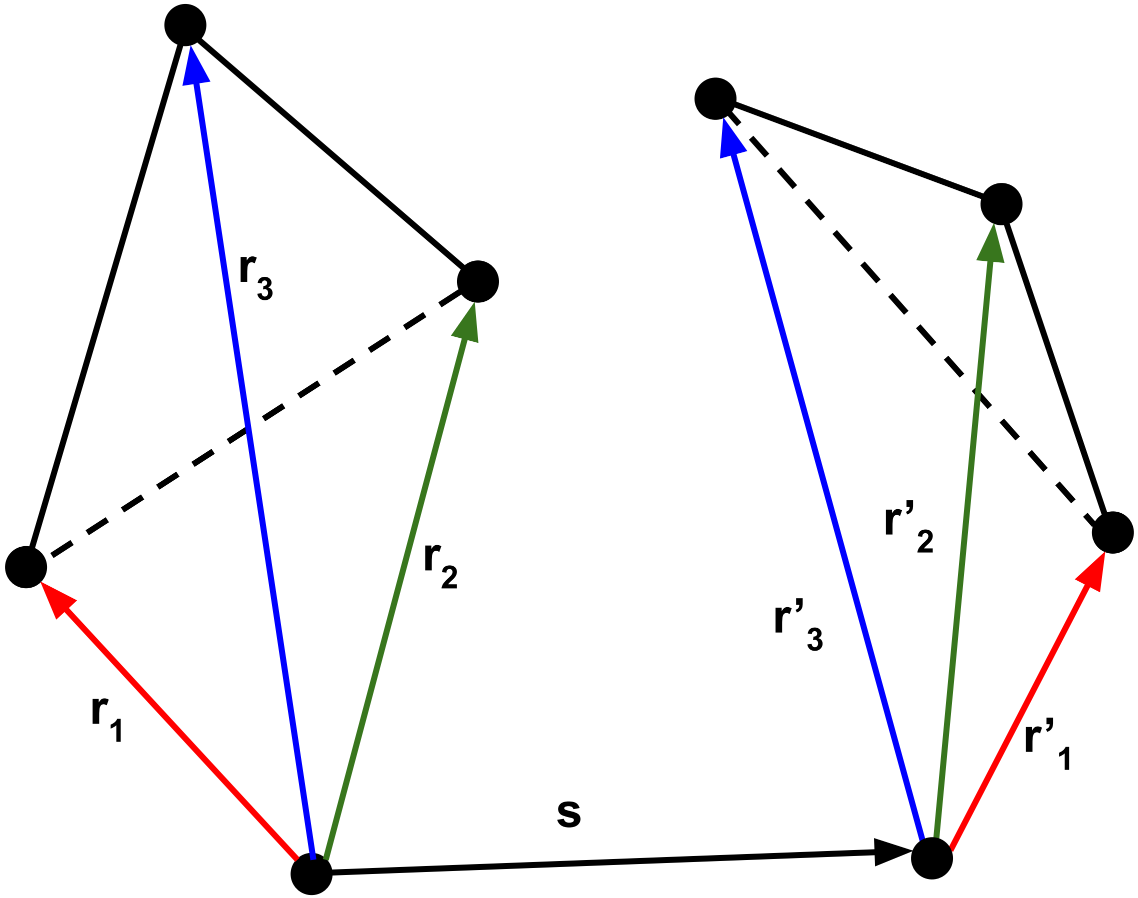
$$W_{\ell,\ell',\ell''}(r_i, r'_j) \equiv \frac{1}{2 r_i r'_j} i^{\ell+\ell'-\ell''} \sqrt{2\ell''+1} r_i^{\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix}^{-1} \sum_{\mathcal{L}=0}^{\ell''} \begin{pmatrix} 2\ell'' \\ 2\mathcal{L} \end{pmatrix}^{1/2} \left(\frac{r'_j}{r_i}\right)^{\mathcal{L}} \sum_m (2m+1)$$

$$\times \begin{pmatrix} \ell & \ell''-\mathcal{L} & m \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell' & \mathcal{L} & m \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \ell & \ell' & \ell'' \\ \mathcal{L} & \ell''-\mathcal{L} & m \end{Bmatrix} 2^{-m} \sum_{a=0}^{\lfloor m/2 \rfloor} (-1)^a \begin{pmatrix} m \\ a \end{pmatrix}$$

$$\times \begin{pmatrix} 2(m-a) \\ m \end{pmatrix} (2 r_i r'_j)^{2a-m} \sum_{b=0}^{m-2a} \begin{pmatrix} m-2a \\ b \end{pmatrix} (r_i^2 + r'^2_j)^{m-2a-b} (-1)^b$$

$$f_{\ell,\ell',\ell''}(r_i, r'_j, s) = \frac{1}{2\pi^2} \left[A I^{[3,\mathrm{lin}]}_{\ell,\ell',\ell''}(r_i, r'_j, s) + \frac{\pi}{2\bar{n}} W_{\ell,\ell',\ell''}(r_i, r'_j) s^{2b-\ell''-1} \right]$$

4PCF

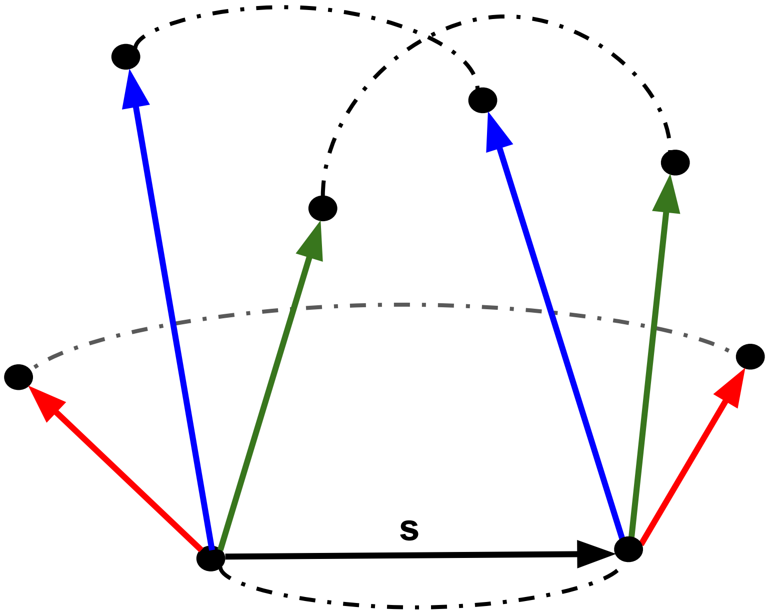


Analytic Covariance Matrix for 4PCF: Case I

$$\mathrm{Cov}^{(\mathrm{fc}),\mathrm{I}}_{\Lambda,\Lambda'}(R,R') = (4\pi)^4 \sum_G (-1)^{\Sigma(\Lambda)(1-\mathcal{E}_G)/2} \sum_{L_1 L_2 L_3} \mathcal{D}^{\mathrm{P}}_{L_1 L_2 L_3}\, \mathcal{C}^{L_1 L_2 L_3}_{000} \begin{Bmatrix} \ell_{G1} & \ell_{G2} & \ell_{G3} \\ \ell'_1 & \ell'_2 & \ell'_3 \\ L_1 & L_2 & L_3 \end{Bmatrix}$$

$$\times \prod_{i=1}^{3} \left[(-1)^{(-\ell_{Gi}-\ell'_i+L_i)/2}\, \mathcal{D}^{\mathrm{P}}_{\ell_i \ell'_i L_i}\, \mathcal{C}^{\ell_{Gi}\ell'_i L_i}_{000} \boxed{\int \frac{s^2 ds}{V} \xi(s) f_{\ell_{Gi}\ell'_i L_i}(r_{Gi}, r'_i, s)} \right]$$

Hou et al. 2021



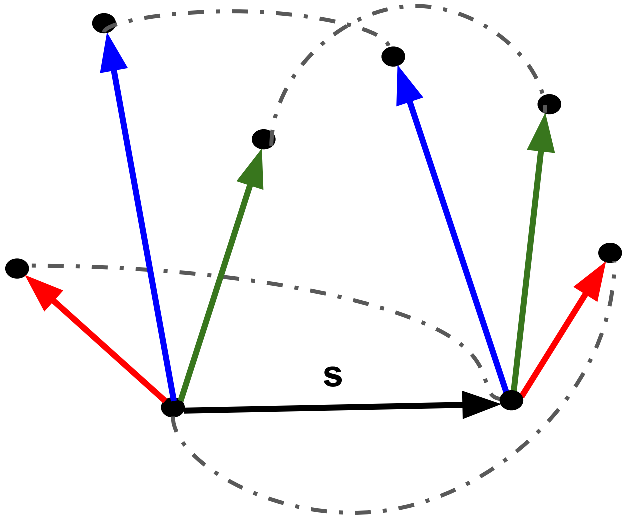
Analytic Covariance Matrix for 4PCF: Case II

$$
\mathrm{Cov}^{(\mathrm{fc}),\mathrm{II}}_{\Lambda,\Lambda'}(R,R') = (4\pi)^4 \sum_{G,H} (-1)^{\Sigma(\Lambda')(1-\mathcal{E}_H)/2} \sum_{L_1 L_2 L_3} \mathcal{D}^{\mathrm{P}}_{L_1 L_2 L_3} \mathcal{C}^{L_1 L_2 L_3}_{000} \begin{Bmatrix} \ell_{G1} & \ell_{G2} & \ell_{G3} \\ \ell'_{H1} & \ell'_{H2} & \ell'_{H3} \\ L_1 & L_2 & L_3 \end{Bmatrix}
$$

$$
\times \prod_{i=1}^{3} \left[(-1)^{(-\ell_{Gi}-\ell'_{Hi}+L_i)/2} \mathcal{D}^{\mathrm{P}}_{\ell_{Gi}\ell'_{Hi}L_i} \mathcal{C}^{\ell_{Gi}\ell'_{Hi}L_i}_{000} \right]
$$

$$
\times \int \frac{s^2 ds}{V} f_{\ell_{G1}0\ell_{G1}}(r_{G1},0,s) f_{0\ell'_{H1}\ell'_{H1}}(0,r'_{H1},s) f_{\ell_{G2}\ell'_{H2}L_2}(r_{G2},r'_{H2},s) f_{\ell_{G3}\ell'_{H3}L_3}(r_{G3},r'_{H3},s)
$$

Hou et al. 2021



Summary

Completed Work

- Accurately modeled P(k)
- Numerically determined covariance of 2PCF
- Determined analytic covariance of 2PCF from P(k) model

Future Work

- Finish solving f-integrals
- Determine analytic covariance of 3PCF and 4PCF from P(k) model