Cosmological Perturbations out of the Box

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Introduction and Summary

It is a standard assumption of many cosmology work that the perturbations on the cosmological background vanish at infinity. A crucial place where this assumption is used is the SVT decomposition. Taking up a spherical topology, for example, a vector on $\mathbb{R}^3$ decomposes uniquely as

$$V^i = \nabla^i \alpha + \left(\vec{\nabla} \times \eta\right)^i$$

This assumption, thus the decomposition, no longer holds if one is interested in non-vanishing behavior. An example of this can be asymptotic symmetry discussions where fields do not vanish trivially, or any scenario one would like to consider a patching of the space.

In such cases one can study the perturbations on **manifolds with boundary** instead of a manifold without a boundary to allow for more general boundary conditions, and for anti-symmetric tensors i.e. forms one can use what is known as **"Hodge-Morrey" decomposition** to perform an analogue decomposition. For example in this case a vector is composed uniquely as

$$V^i = \nabla^i \alpha + \left(\vec{\nabla} \times \eta\right)^i + \nabla^i \sigma$$

where

$$\alpha|_{\partial M} = 0\,, \quad \eta^{\perp}\Big|_{\partial M} = 0\,, \quad \nabla^2 \sigma = 0\ .$$

We note that the harmonic part can be absorbed into gradient and curl parts if boundary conditions are to be removed.

Thus **for a case with non-vanishing boundary conditions decomposition into gradient and curl is not unique, and thus cannot be used e.g. for decomposition of equations of motions of cosmological perturbations.**

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However for most perturbative gravity/cosmology discussions we need the decomposition of the (spatial part) of the metric tensor. Using Hodge-Morrey decomposition and following a similar strategy to that of Straumann [1], in the following we prove a decomposition theorem for symmetric rank-2 traceless tensors on Ricci flat manifolds.

We will show how this decomposition is to be used in cosmological perturbation theory, and how some familiar scenarios change with the new decomposition.

[1] Norbert Straumann. Proof of a decomposition theorem for symmetric tensors on spaces with constant curvature. Annalen Phys., 17:609–611, 1997.

Manifolds with Boundary

A Manifold with boundary is defined in a similar way to a manifold, but in a way that allows for vectors off-the boundary. Let us consider such a manifold called M with the boundary ∂M. We will define a metric on it and a corresponding induced metric the boundary. On the boundary we will decompose any tensor ω into $\mathbf{t}\omega$ and $\mathbf{n}\omega$ so that

$$\mathbf{t}\omega\left(X_1,\cdots X_p\right)=\omega\left(X_1^{\parallel},\cdots X_p^{\parallel}\right)\quad \forall X_i\in \left.TM\right|_{\partial M}\ ,$$

and

$$\mathbf{n}\omega:=\left.\omega\right|_{\partial M}-\mathbf{t}\omega\ .$$

With these definitions, then one has the useful identities

$$*\,\mathbf{n}\omega=\mathbf{t}*\omega\quad\text{and}\quad *\,\mathbf{t}\omega=\mathbf{n}*\omega$$
$$\mathbf{t}d\omega=d\mathbf{t}\omega\quad\text{and}\quad \mathbf{n}d^{\dagger}\omega=d^{\dagger}\mathbf{n}\omega\ ,$$

where

$$d^{\dagger}\omega=(-1)^{nk+n+1}*d*\omega\ .$$

Hodge-Morrey decomposition

On manifolds with boundary in place of harmonics forms (forms that satisfy $(dd^\dagger + d^\dagger d)\kappa = 0$), it is more useful to define harmonic fields: forms that satisfy $d\kappa = 0$ and $d^\dagger \kappa = 0$.

On a manifold with trivial cohomology this means κ is both exact ($\kappa = d\alpha$ for some α) and coexact ($\kappa = d^\dagger \beta$ for some β) (except for rank 0, where κ is a constant). Now let us state the decomposition theorem:

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Theorem (Hodge-Morrey Decomposition[2])

Any square integrable k-form on a compact ∂-manifold M can be uniquely written as

$$w = d\alpha + d^\dagger \beta + \kappa$$

where $\kappa \in \mathcal{H}^k(M)$, i.e. is a harmonic field, $\mathbf{t}\alpha = 0, \mathbf{n}\beta = 0$.

[2] Günter Schwarz. Hodge decomposition : a method for solving boundary value problems. Springer, 1995.

A decomposition theorem for rank-2 symmetric tensors

Now we proceed to decomposition of rank-2 symmetric tensors. In a similar vein with the argument of Straumann, we start with the decomposition of the divergence of the tensor. As a one form it has a unique decomposition

$$\nabla^i t_{ij} = \nabla_j \alpha + d^\dagger \beta_j + \kappa_j$$

where $\mathbf{t}\alpha = 0$, $\mathbf{n}\beta = 0$ and κ is a harmonic field. We will try to bring right hand side to the $\nabla^i t_{ij}$ form to find the decomposition.Now the argument is, given $d^\dagger \beta$ one can find a unique one-form t^{nd} such that

$$d^\dagger \beta = d^\dagger d t^{nd} \quad \text{and} \quad \mathbf{n} d t^{nd} = 0\,, \quad \mathbf{n} t^{nd} = 0\,, \quad d^\dagger t^{nd} = 0\ .$$

This can be proved using the BVP theorems in Schwarz' book. Now for a Ricci flat geometry one can then show that

$$(d^\dagger \beta)_j = (d^\dagger d t^{nd})_j = \nabla^i \nabla_{[i} t^{nd}_{j]} = \nabla^i \left(\nabla_{(i} t^{nd}_{j)} - g_{ij} \nabla \cdot t^{nd} \right) - R^i_j t^{nd}_i = \nabla^i (\nabla_{(i} t^{nd}_{j)})\ .$$

This brings the $d^\dagger \beta$ part to the desired form.

For the remaining parts, we make the following observation: for trivial cohomology $\kappa = d\sigma$ for some σ. Then we note for a given scalar $\alpha + \sigma$ there exists a unique t^t such that

$$\nabla^2 t^t = \frac{m}{m-1}(\alpha + \sigma) \quad \text{and} \quad \mathbf{t}t^t = 0.$$

This follows again from the BVP theorems in Schwarz' book. Given $\nabla^i t_{ij}$, $d(\alpha + \sigma)$ is fixed, thus $\alpha + \sigma$ is fixed up to a constant. However for a Ricci flat case

$$\nabla^i \diamondsuit_{ij} t^t = \nabla^i \left(\nabla_{(i}\nabla_{j)} t^t - \frac{g_{ij}}{m} \nabla^2 t^t \right) = \frac{(m-1)}{m} \nabla_j \nabla^2 t^t,$$

thus the part coming from this constant does not contribute to the expression $\nabla^i \diamondsuit_{ij} t^t$, and this expression is what we need for the decomposition.

Using the conclusions we have arrived above, we can now state the decomposition theorem:

Theorem

Any symmetric traceless rank-2 tensor t_{ij} on an arbitrary dimensional ∂-manifold with Ricci flat metric can be uniquely decomposed as

$$t_{ij} = \diamondsuit_{ij} t^{t} + \nabla_{(i} t^{nd}_{j)} + t^{TT}_{ij} \ ,$$

where

$$\mathbf{t} t^{t} = 0\,, \quad \nabla^{i} t^{nd}_{i} = 0\ , \mathbf{n} t^{nd} = 0\ , \mathbf{n} d t^{nd} = 0\,, \quad \nabla^{i} t^{TT}_{ij} = 0\ , g^{ij} t^{TT}_{ij} = 0.$$

Further decompositions

Unfortunately decomposition we have just found will not be enough for applications to cosmological perturbation theory. We will need two more decompositions. We begin with the first one: In perturbative equations one will have mixing of the terms of the form

$$\nabla_i \nabla_j \sigma + t_{ij}^{TT}$$

where $\nabla^2 \sigma = 0$. Note that $\nabla_i \nabla_j \sigma$ is TT and thus cannot be separated from any other t_{ij}^{TT} type of tensor with our new decomposition.To separate these, using a Taylor expansion one can show that:

Theorem

In the flat $\mathbb{R}^m$ space any rank-2 symmetric TT tensor V can be decomposed as

$$V_{ij} = X_{ij} + \partial_i \partial_j \sigma$$

where $\partial^2 \sigma = 0$ and X is a rank-2 symmetric TT tensor that satisfies

$$\partial_{(i} X_{jk)}(x) = 0 \quad \forall i, j, k \quad \textit{and} \quad \forall x \in \mathbb{R}^m \ .$$

We will denote this last part as X_{ij}^{TTA}.

Secondly, if we decompose a vector v using the Hodge-Morrey decomposition

$$v = d(v^t + v^h) + v^n$$

where $d^\dagger v^n = 0$ and $\mathbf{n}v^n = 0$,in the cosmological perturbation equations, we will have expressions such as

$$\partial_{(i}v_{j)} = \partial_i\partial_j v^t + \partial_i\partial_j v^h + \partial_{(i}v^n_{j)} \ .$$

First two parts are of one of the tensorial types we have discussed above, but the last part is not: it needs the extra condition $\mathbf{n}dv^n = 0$ to be so.

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First two parts are of one of the tensorial types we have discussed above, but the last part is not: it needs the extra condition $\mathbf{n} d v^n = 0$ to be so. Thus we need to decompose this part further. One can show that it is possible to write

$$\partial_{(i} v^n_{j)} = \diamondsuit_{ij} v^{th} + \partial_{(i} v^{nd}_{j)} + v^{TT}_{ij}$$

where

$$\mathbf{t} v^{th} = 0 \ , \partial^2 \partial^2 v^{th} = 0 \, , \quad d^\dagger v^n = 0 \ , \mathbf{n} v^n = 0 \ , \mathbf{n} d v^n = 0 \ .$$

Cosmological Perturbation Theory Revisited

Now we apply the decomposition we have described to the cosmological perturbations. We will consider the linearized Einstein equations for the metric

$$ds^2 = -(1+2\Phi)dt^2 + 2a(t)N_i dx^i dt + a^2(t)\left((1-2\Psi)\delta_{ij} + 2\gamma_{ij}\right) dx^i dx^j \ ,$$

FRW spacetime and perturbations.Each of the perturbative fields here are either a scalar or a vector or a tensor on $\mathbb{R}^3$. We will compose each of these as follows:

- scalars such as Φ will decompose as

$$\Phi = \Phi^t + \Phi^h \quad \text{where} \quad \mathbf{t}\Phi^t = 0 \quad \text{and} \quad d^\dagger d\Phi^h = 0 \ .$$

  That this is a unique decomposition can be easily shown.
- the vector N_i will decompose by the Hodge Morrey theorem:

$$N = d\psi + N^n = d(\psi^t + \psi^h) + N^n$$

  where

$$\mathbf{t}\psi^t = 0\,, \quad d^\dagger d\psi^h = 0\,, \quad d^\dagger N^n = 0 \quad \text{where} \quad \mathbf{n}N^n = 0 \ .$$

Whenever need we will also use

$$\partial_{(i}N_{j)} = \partial_i\partial_j\psi + \partial_{(i}N^n_{j)} = \frac{g_{ij}}{m}\partial^2\psi + \diamondsuit_{ij}(\psi^t + N^{th}) + \partial_{(i}N^{nd}_{j)} + \partial_i\partial_j(\psi^h + N^h) + N^{TTA}_{ij}$$

- the tensor γ_{ij} will be decomposed as

$$\gamma_{ij} = \diamondsuit_{ij}\gamma^t + \partial_{(i}\gamma^{nd}_{j)} + \nabla_i\nabla_j\gamma^h + \gamma^{TTA}_{ij} \ ,$$

each part satisfying the relevant conditions given by the decomposition theorems.

Using this, perturbative Einstein equations decompose as

$$6H^2\Phi + 6H\dot{\Psi} - 2\frac{\partial^2\Psi}{a^2} - \frac{2}{3a^2}\partial^2\partial^2\gamma^t + 2H\frac{\partial^2\psi}{a} + \rho = 0 \ ,$$

$$\partial_i(H\Phi + \dot{\Psi} + \frac{1}{3}\partial^2\dot{\gamma}^t - \frac{1}{3a}\partial^2 N^{th} - \dot{H}v) = 0 \ ,$$

$$(\partial_t + H)\left[3a^2(\dot{\Psi} + H\Phi) + \partial^2(a\psi)\right] + \partial^2(\Phi - \Psi) - \frac{1}{3}\partial^2\partial^2\gamma^t = \frac{3a^2}{2}(p - 2\dot{H}\Phi) \ ,$$

$$\diamondsuit_{ij}\left[-\Phi^t + \Psi^t + \frac{(\partial^2\gamma^t)^t}{3} + (\partial_t + H)\left(a^2\dot{\gamma}^t - a\psi^t - aN^{th}\right) - a^2\Sigma^t\right] = 0 \ ,$$

$$\nabla_i\nabla_j\left((\Psi - \Phi)^h + \frac{(\partial^2\gamma)^h}{3} + (\partial_t + H)\left(a^2\dot{\gamma}^h - aN^h - a\psi^h\right) - a^2\Sigma^h\right) = 0 \ ,$$

$$\frac{1}{2}\partial^2\dot{\gamma}_j^{nd} - \frac{1}{2a}\partial^2 N_j^{nd} - 2\dot{H}v_i^n = 0 \ ,$$

$$(\partial_t + H)\left[a^2\partial_{(i}\dot{\gamma}_{j)}^{nd} - a\partial_{(i}N_{j)}^{nd}\right] - a^2\partial_{(i}\Sigma_{j)}^{nd} = 0 \ ,$$

$$(\partial_t + H)\left(a^2\gamma_{ij}^{TTA} - aN_{ij}^{TTA}\right) - a^2\Sigma_{ij}^{TTA} = 0 \ .$$

Example: Single Field Inflation

Let us now see how some standard cosmological scenario will work out with the new decomposition/non-vanishing boundary condition.

We will consider the single field inflation with the field $\bar{\varphi} + \varphi$. We will focus on the scalar equations of motion.

We choose a gauge where

$$\varphi = 0 \ , \partial_i \partial_j N^h = 0 \ , \partial_i \partial_j \gamma^h = 0 \ , \diamondsuit_{ij} N^{th} = 0 \ , \diamondsuit_{ij} \gamma^t = 0 \ .$$

Then we are left with scalars Φ, Ψ each of which has a t-type and h-type parts and ψ which is of t-type. Equations of motion we are left with are

$$\begin{aligned}
&(3H^2 + \dot{H})\Phi + 3H\dot{\Psi} - \frac{\partial^2 \Psi}{a^2} + H\frac{\partial^2 \psi}{a} = 0 \ , \\
&\partial_i \left(H\Phi + \dot{\Psi} \right) = 0 \ , \\
&\dot{H}\dot{\Phi} + (\ddot{H} + 6H\dot{H})\Phi + \dot{H}\left(3\dot{\Psi} + \frac{\partial^2 \psi}{a} \right) = 0 \ .
\end{aligned}$$

Second equation can be written as

$$H\Phi^t + \dot{\Psi}^t = 0 \quad \text{and} \quad H\Phi^h + \dot{\Psi}^h = A(t)$$

where $A(t)$ is an arbitrary space-independent field. Using this and other two equations above one can write

$$\ddot{\Psi}^t + \left(\frac{\ddot{H}}{\dot{H}} - 2\frac{\dot{H}}{H} + 3H\right)\dot{\Psi}^t - \frac{(\partial^2\Psi)^t}{a^2} = 0 \; ,$$

$$(\ddot{\Psi}^h - \dot{A}) + \left(\frac{\ddot{H}}{\dot{H}} - 2\frac{\dot{H}}{H} + 3H\right)(\dot{\Psi}^h - A) - \frac{(\partial^2\Psi)^h}{a^2} = 0 \; .$$

These are two copies of the Mukhanov-Sasaki equation. Note that Ψ^t piece works exactly the same as the standard case with vanishing boundary, whereas the harmonic part Ψ^h couples with the arbitrary field $A(t)$. This arbitrary part is expected to be fixed by the boundary conditions.

- We have proved a new decomposition theorem for symmetric rank two tensors with non-vanishing boundary conditions.
- Using this we have revisited the cosmological perturbation equations.
- For the standard single field inflation scenario, we showed that we have two types of scalars describing the fluctuations, one with vanishing boundary value, another harmonic.
  For further investigation, one needs to impose some physical boundary conditions: either some non-trivial conditions at infinity, or by connecting the bounded region with outside.
  Note that for even solving a equation of motion like Mukhanov-Sasaki, one needs to change the usual method of Fourier transformation with Fourier series expansion since we are in a bounded region.
- Nevertheless, we hope our investigation opens a path to study cosmological (and perhaps other gravitational) perturbations with non standard boundary conditions, and lead to interesting insights.

Thank You!