Stochastic inflation and primordial black holes: challenges and solutions

Cosmology from Home 2023

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Based on 2012.06551, 2111.07437, 2210.17441, 2304.10903
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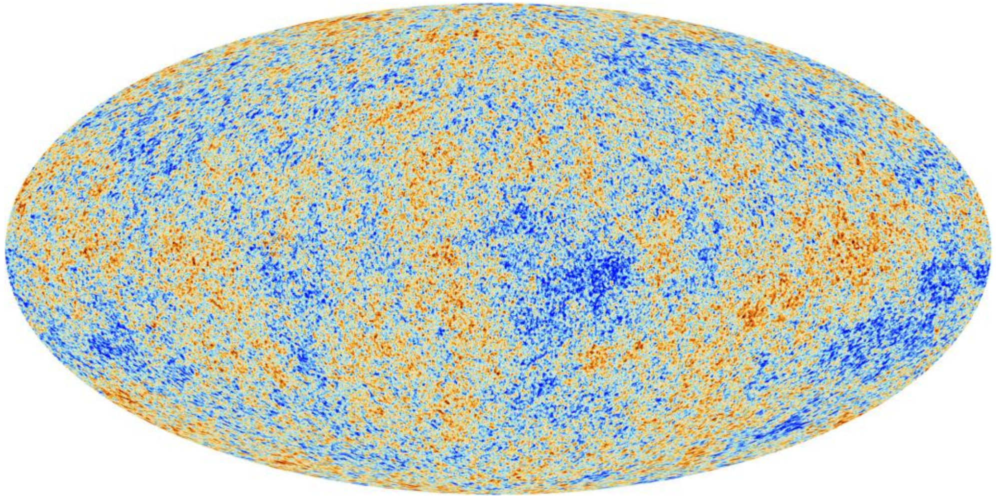
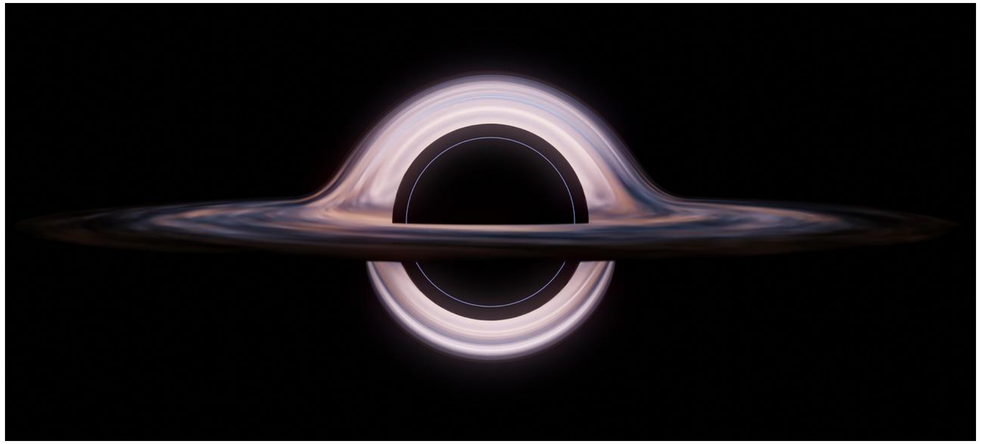
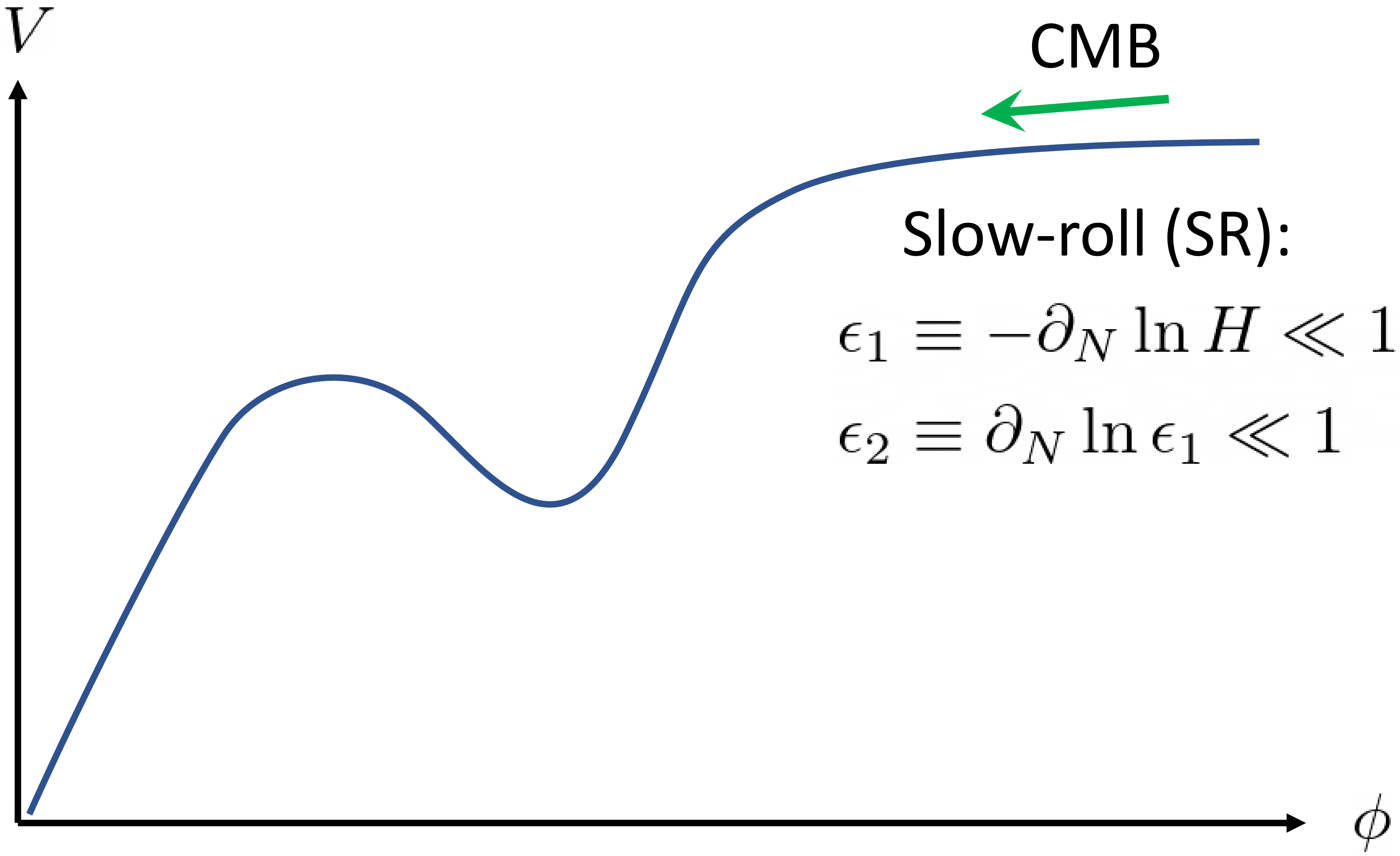
Primordial perturbations

Cosmic inflation: quantum fluctuations

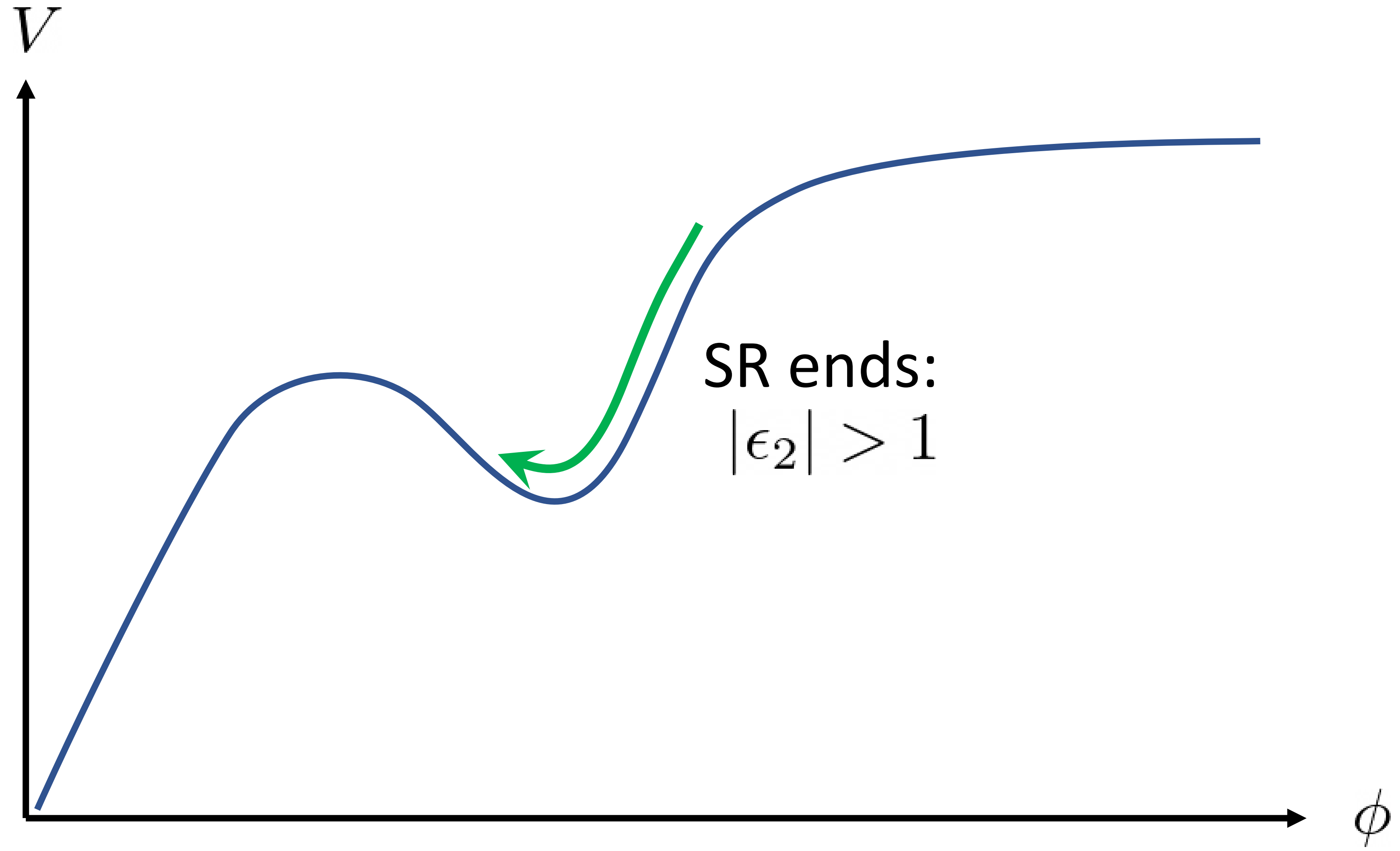
Later: strongest collapse into black holes

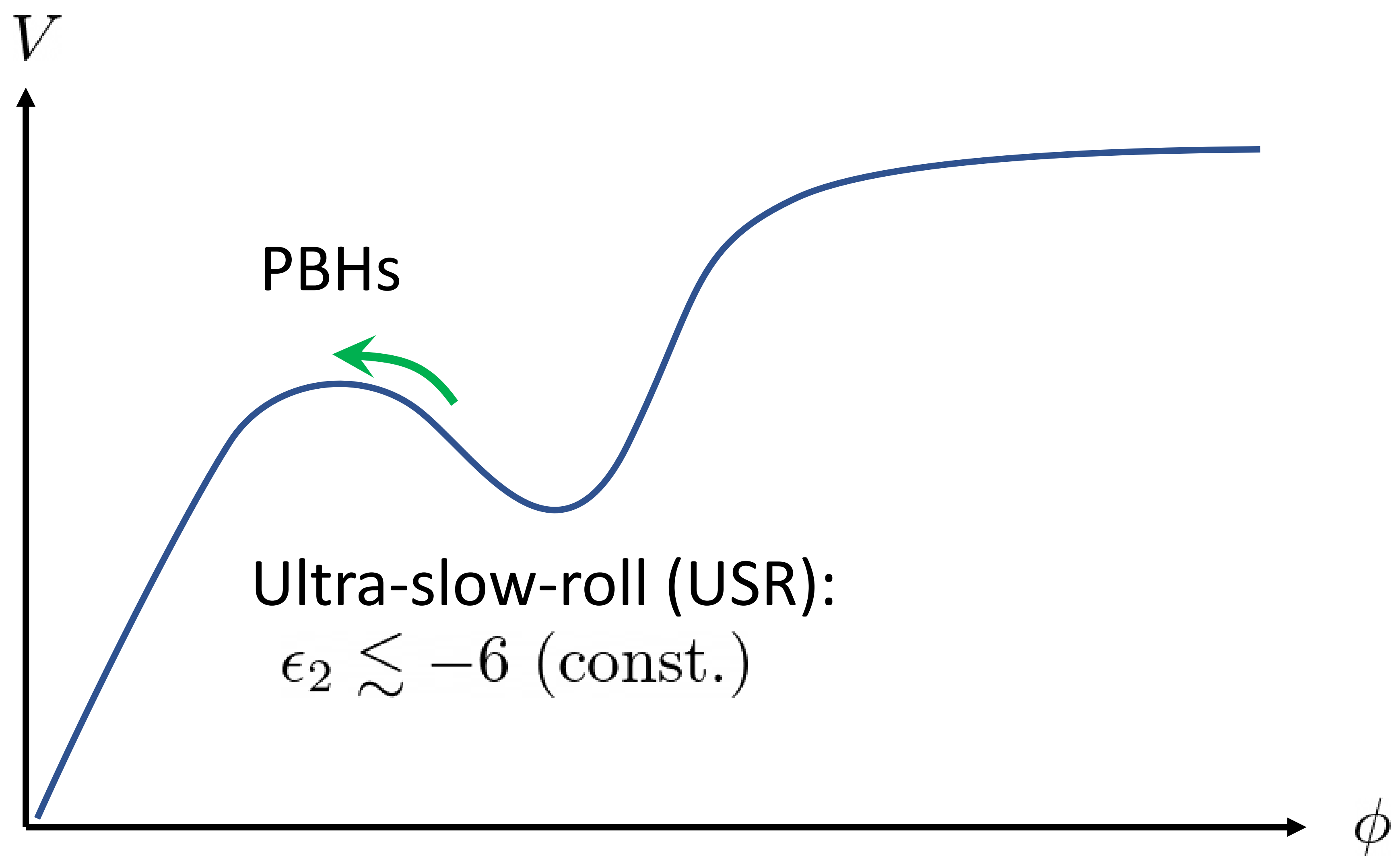


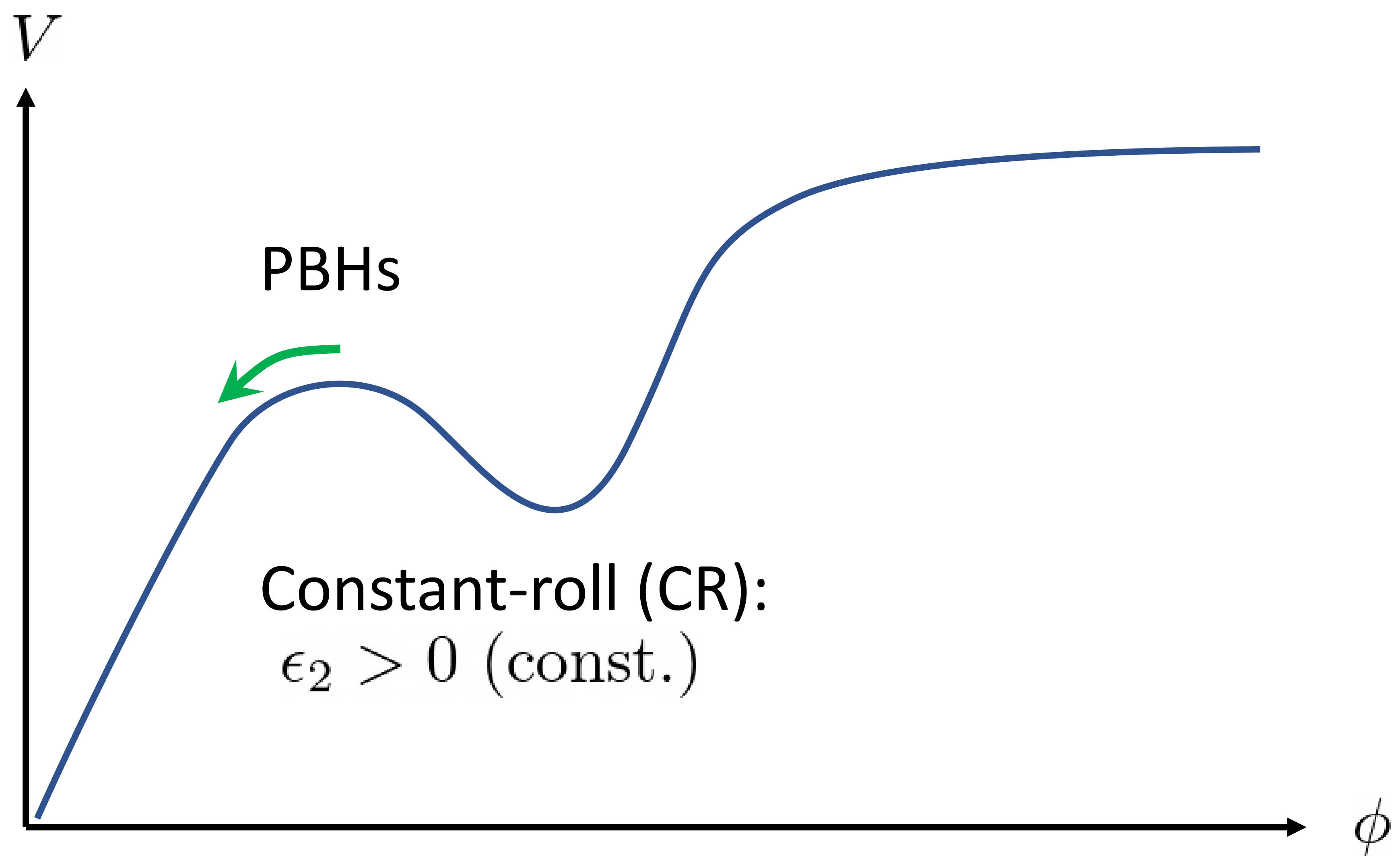




V
CMB
Slow-roll (SR):
$\epsilon_1 \equiv -\partial_N \ln H \ll 1$
$\epsilon_2 \equiv \partial_N \ln \epsilon_1 \ll 1$
ϕ

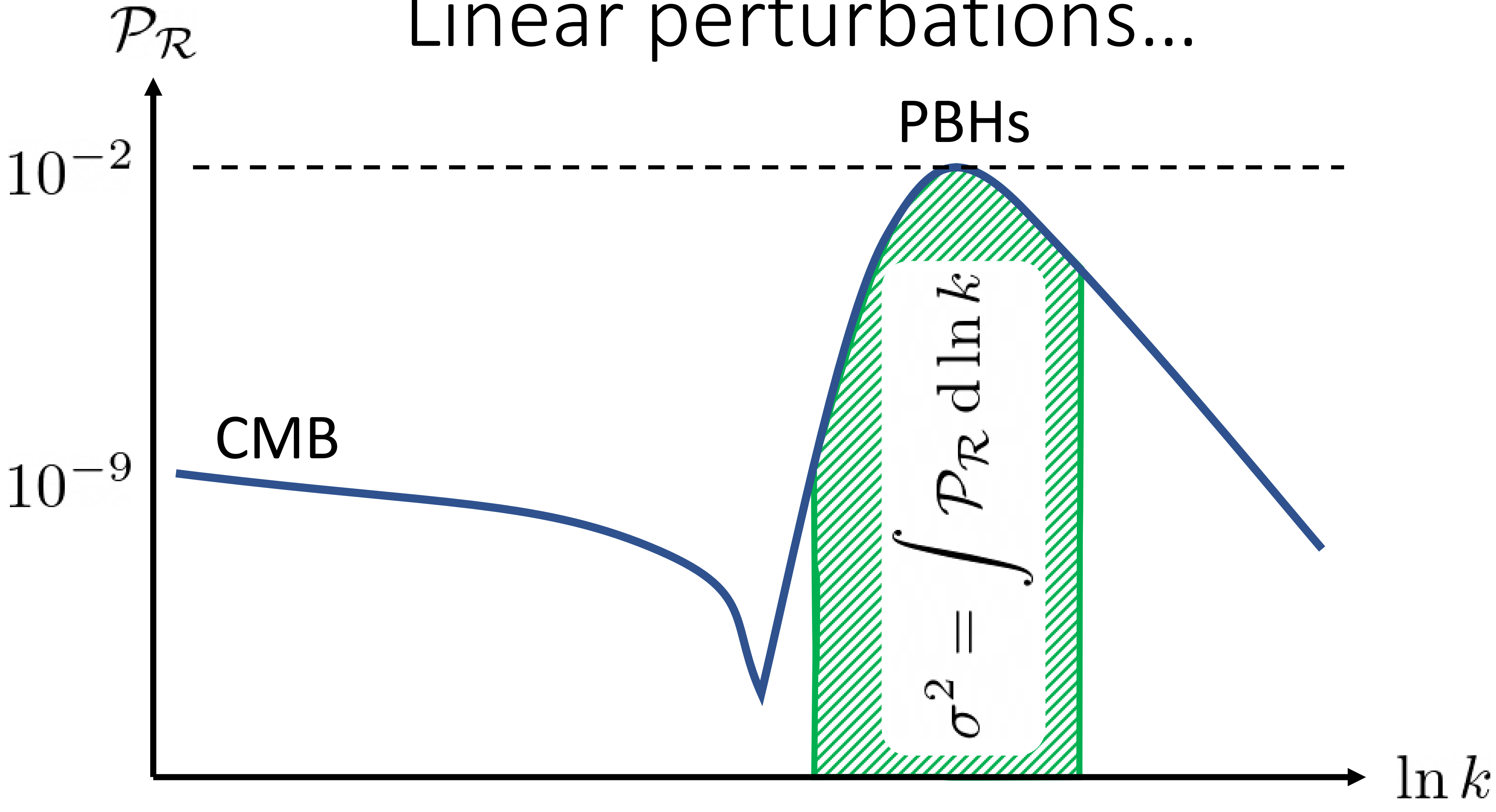

V
ϕ
SR ends:
$|\epsilon_2| > 1$


V
PBHs
Ultra-slow-roll (USR):
$\epsilon_2 \lesssim -6$ (const.)
ϕ

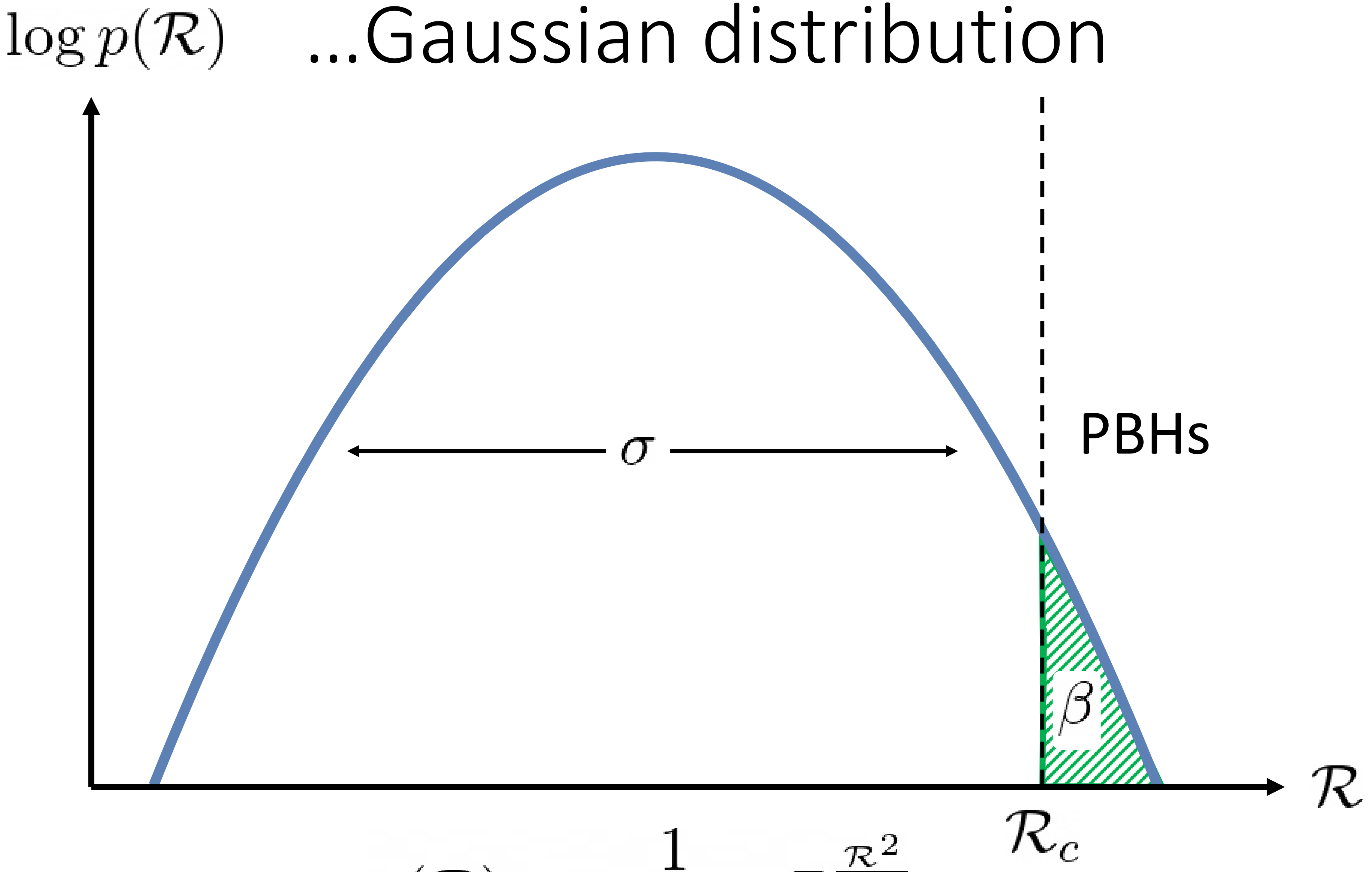


V
PBHs
Constant-roll (CR):
$\epsilon_2 > 0$ (const.)
ϕ

Linear perturbations...



…Gaussian distribution



$$p(\mathcal{R}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\mathcal{R}^2}{2\sigma^2}}$$

Beyond perturbation theory

FLRW evolution is non-linear

Super-Hubble patches ≈ local FLRW universes

$$\mathcal{R} = \Delta N$$

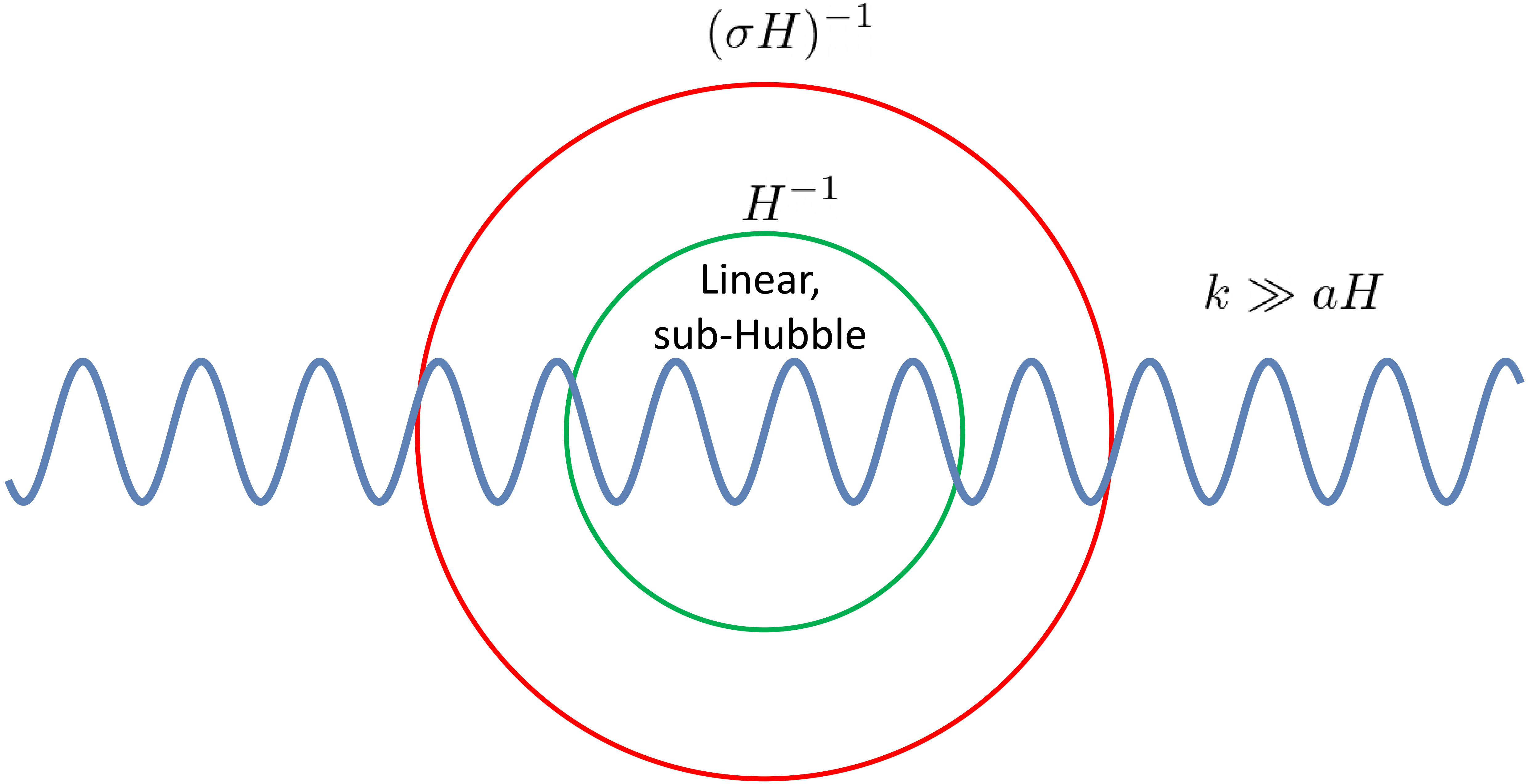
Stochastic inflation

Inflaton field: $\phi + \delta\phi$

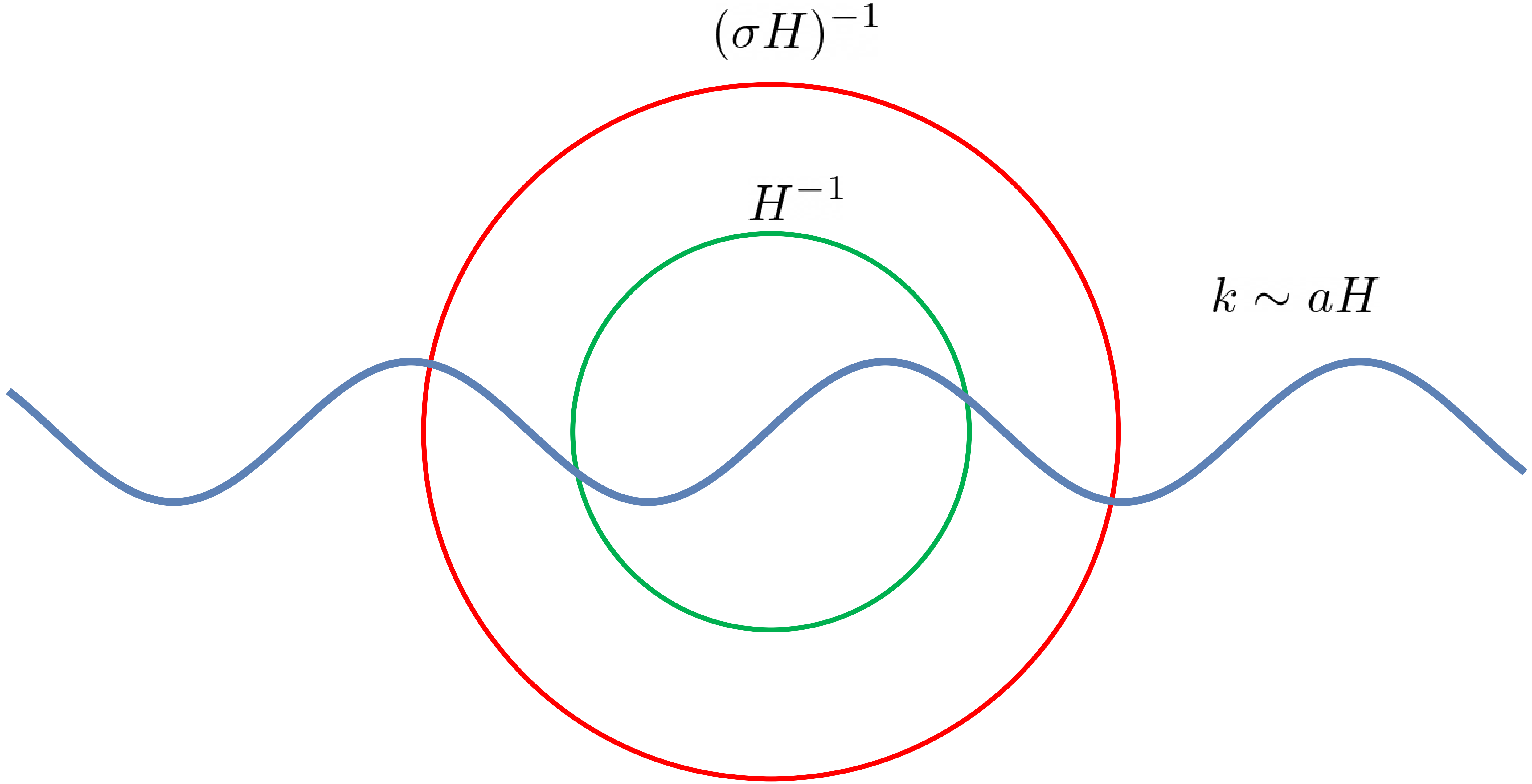
Coarse-grained:
FLRW

Short-wavelength:
linear perturbation theory

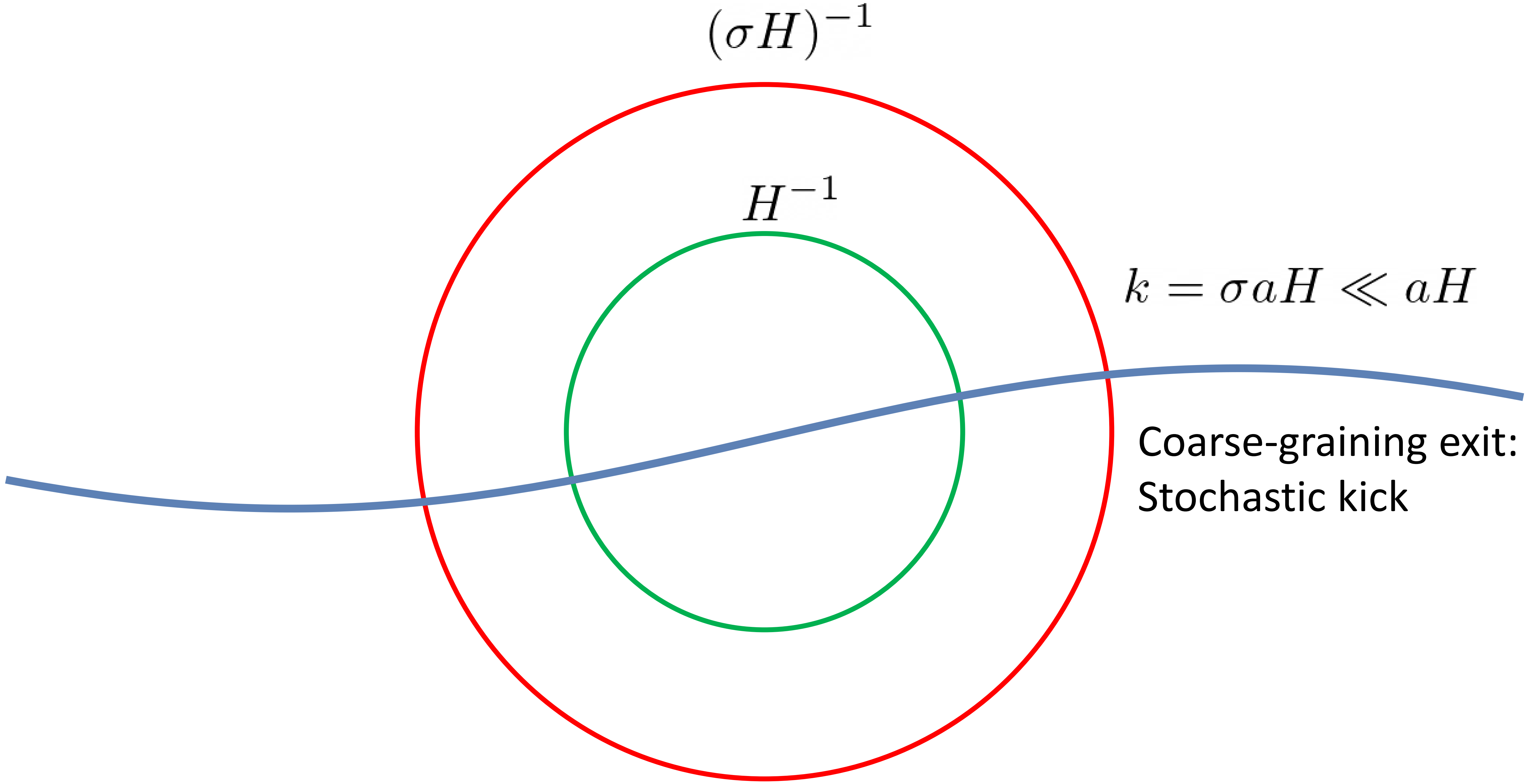
Patched together at the coarse-graining scale $k = k_\sigma \equiv \sigma a H$



$(\sigma H)^{-1}$
H^{-1}
Linear,
sub-Hubble
$k \gg aH$



$(\sigma H)^{-1}$
H^{-1}
$k \sim aH$


$(\sigma H)^{-1}$
H^{-1}
$k = \sigma a H \ll a H$
Coarse-graining exit:
Stochastic kick

Stochastic inflation

$$\phi' = \pi + \xi_\phi\,, \quad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'(\phi)}{H^2} + \xi_\pi\,, \quad H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$

$$\delta\phi''_k = -(3 - \frac{1}{2}\pi^2)\delta\phi'_k - \left[\frac{k^2}{a^2H^2} + \pi^2(3 - \frac{1}{2}\pi^2) + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\right]\delta\phi_k$$

$$\langle\xi_\phi(N)\xi_\phi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi_{k_\sigma}(N)|^2\delta(N - N')$$

$$\langle\xi_\pi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi'_{k_\sigma}(N)|^2\delta(N - N')$$

$$\langle\xi_\phi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}\delta\phi_{k_\sigma}(N)\delta\phi'^*_{k_\sigma}(N)\delta(N - N')$$

$$\mathcal{R} = \Delta N \equiv N - \bar{N}$$

Challenges

Need to solve a coupled set of complicated equations

Need statistics: PBHs from rare realizations

Need accuracy: perturbations sensitive to small changes

Further approximations?

[Pattison et al, 1707.00537]
[Ezquiaga et al, 1805.06731]
[Biagetti et al, 1804.07124]
…

Assume de Sitter behavior for $\delta\phi$: $|\delta\phi_{k_\sigma}|^2 \approx \frac{H^2}{2k_\sigma^3}$

Assume a simplified potential (flat sections, steps, …)

Get analytical solutions. Accuracy?

Numerics?

[Figueroa et al, 2012.06551]
[Figueroa et al, 2111.07437]

Solve with supercomputers (millions of CPU hours)

Only case studies: tuning parameters too costly

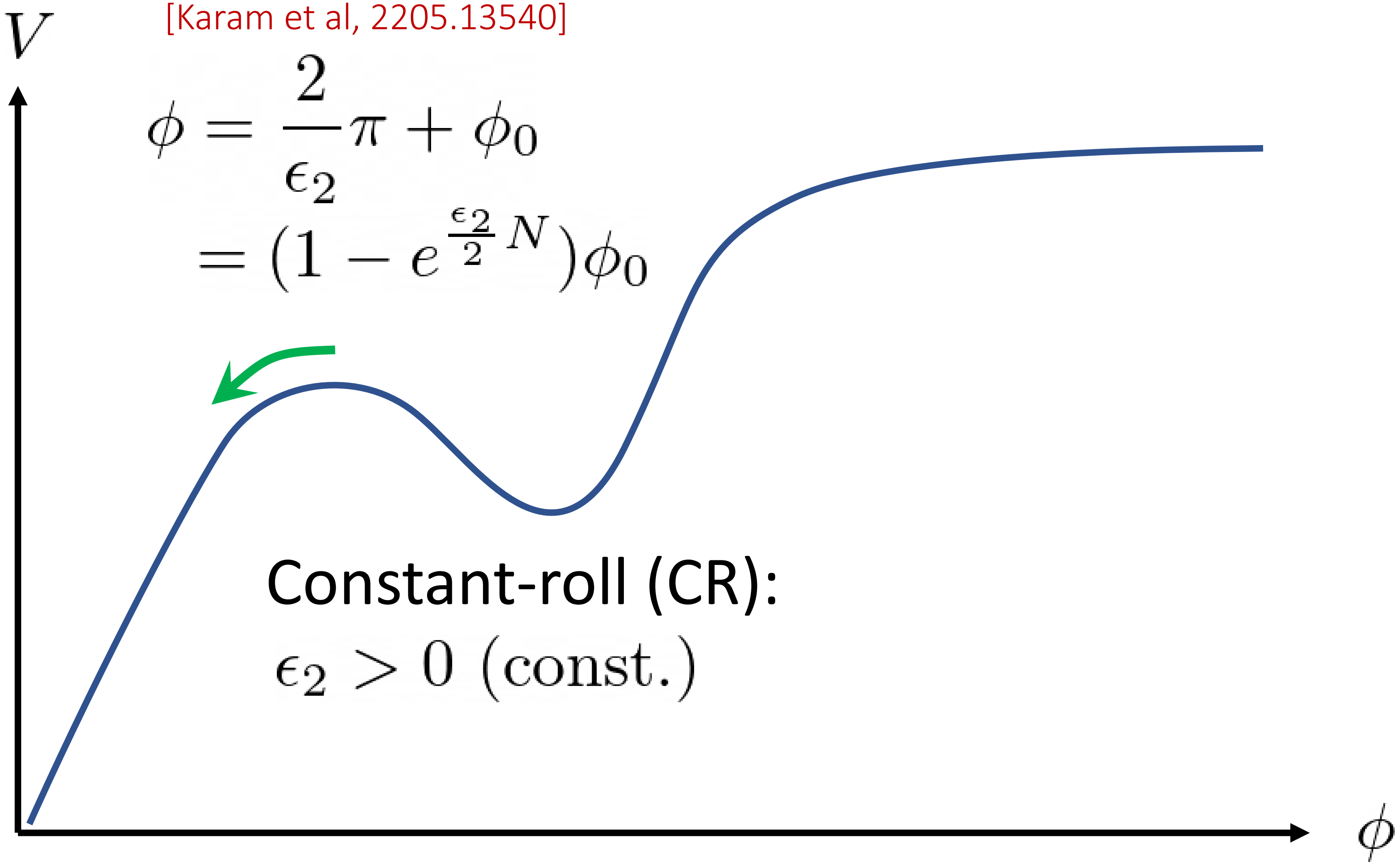
Solution: constant-roll [Tomberg, 2304.10903]

- CR: coarse-graining of strongest modes
- Power spectrum frozen: motion confined to classical trajectory
- Classical trajectory simple

Can solve the system analytically,
for any linear power spectrum

[Karam et al, 2205.13540]

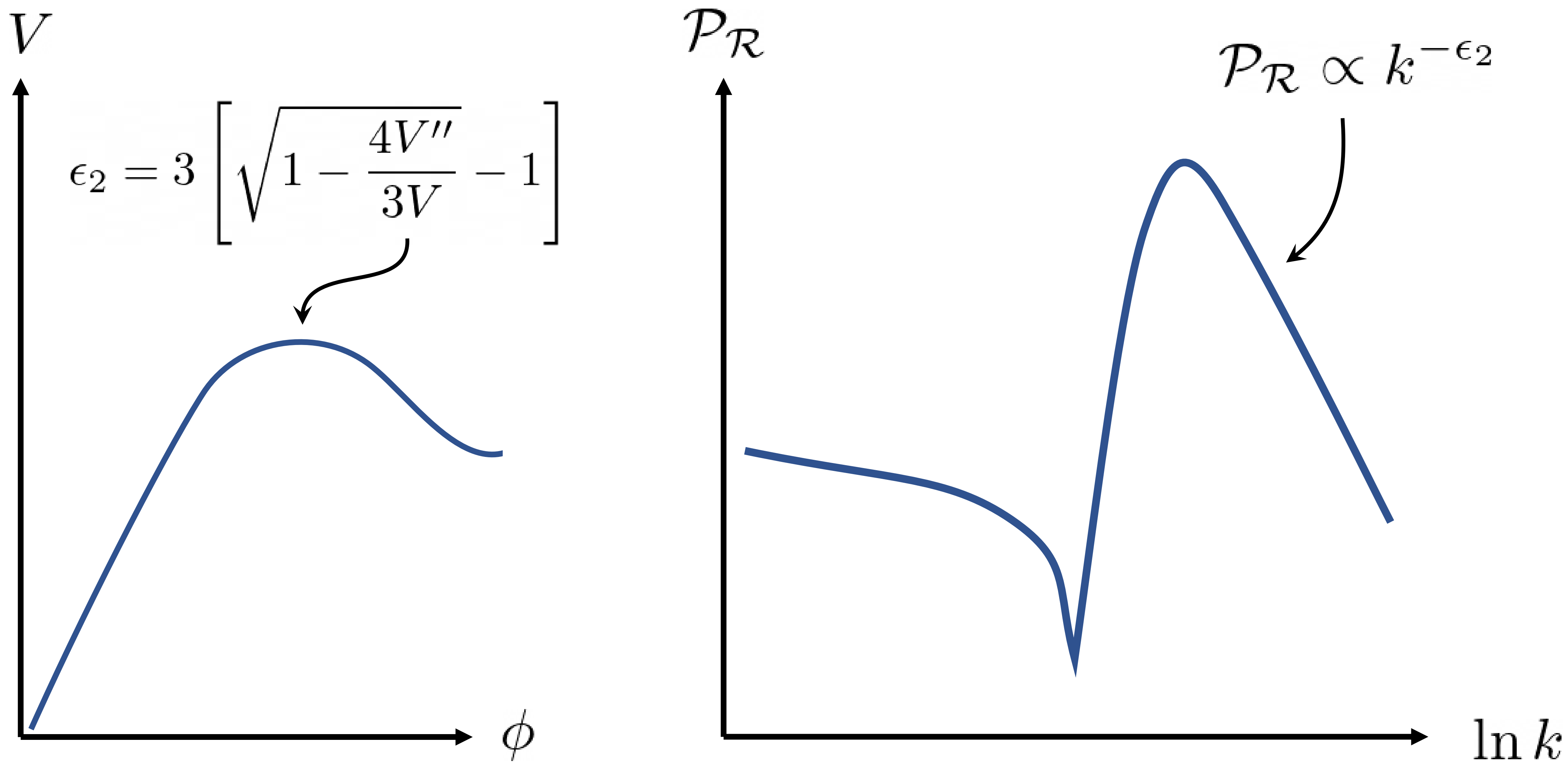
$$\phi = \frac{2}{\epsilon_2}\pi + \phi_0$$
$$= (1 - e^{\frac{\epsilon_2}{2}N})\phi_0$$



Constant-roll (CR):

$\epsilon_2 > 0$ (const.)

[Karam et al, 2205.13540]



Simplified stochastic equation:

$$\mathrm{d}\phi = \frac{\epsilon_2}{2}(\phi + \phi_0)\mathrm{d}N + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N}\sqrt{\mathcal{P}_\mathcal{R}(k_\sigma)\mathrm{d}N}\,\hat{\xi}_N$$

$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

$$\phi(N) = \phi_0\left(1 - e^{\frac{\epsilon_2}{2}N}\right) + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N} X(N)$$

$$X(N) \equiv \sum_{k=k_{\mathrm{ini}}}^{k=k_\sigma(N)} \sqrt{\mathcal{P}_\mathcal{R}(k)\,\mathrm{d}\ln k}\,\hat{\xi}_k$$

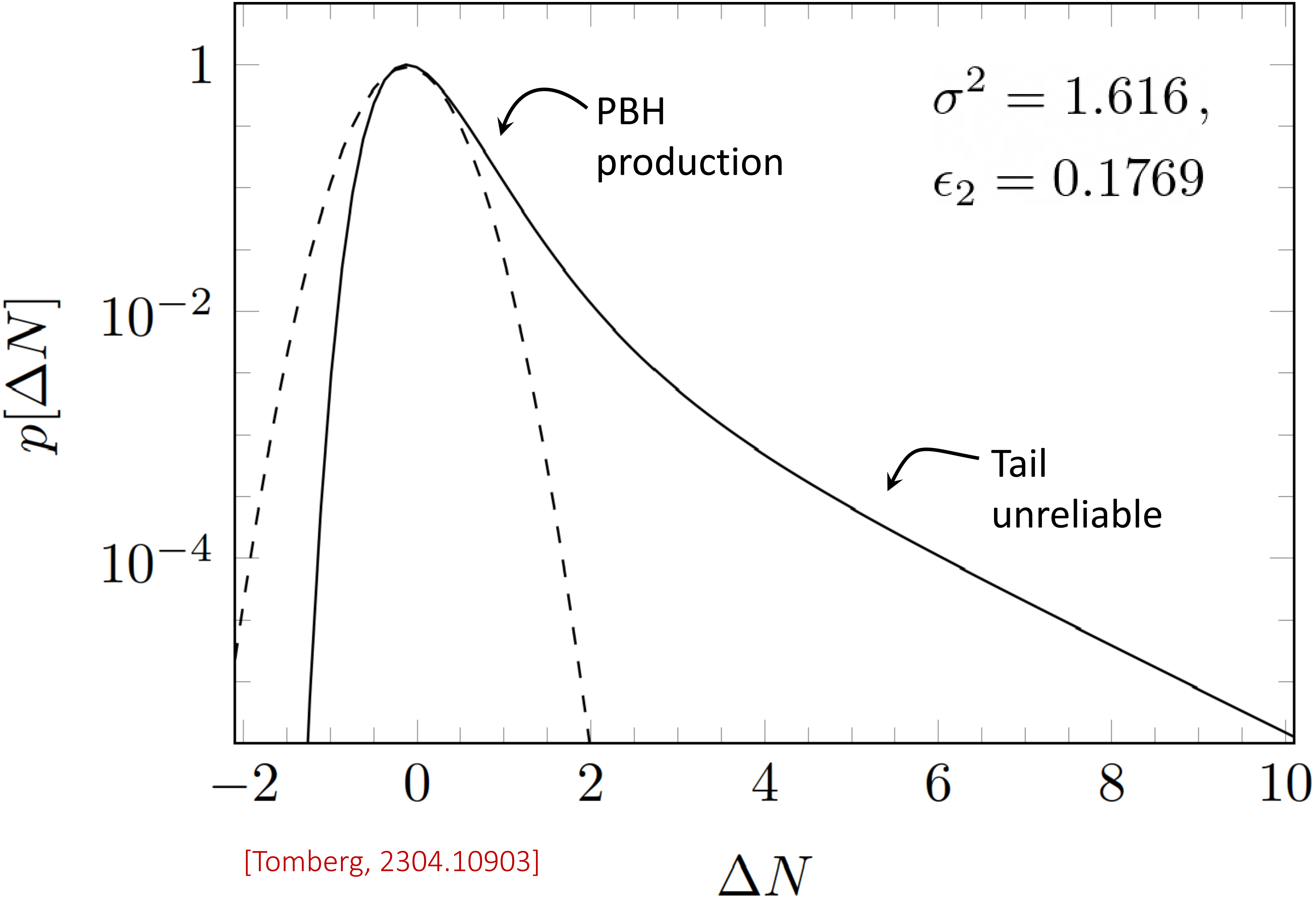
ΔN distribution

$$p(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{X^2}{2\sigma^2}}\,, \qquad \sigma^2 \equiv \int_{k_{\rm ini}}^{k_{\rm end}} \mathcal{P}_{\mathcal{R}}(k)\, \mathrm{d}\ln k$$

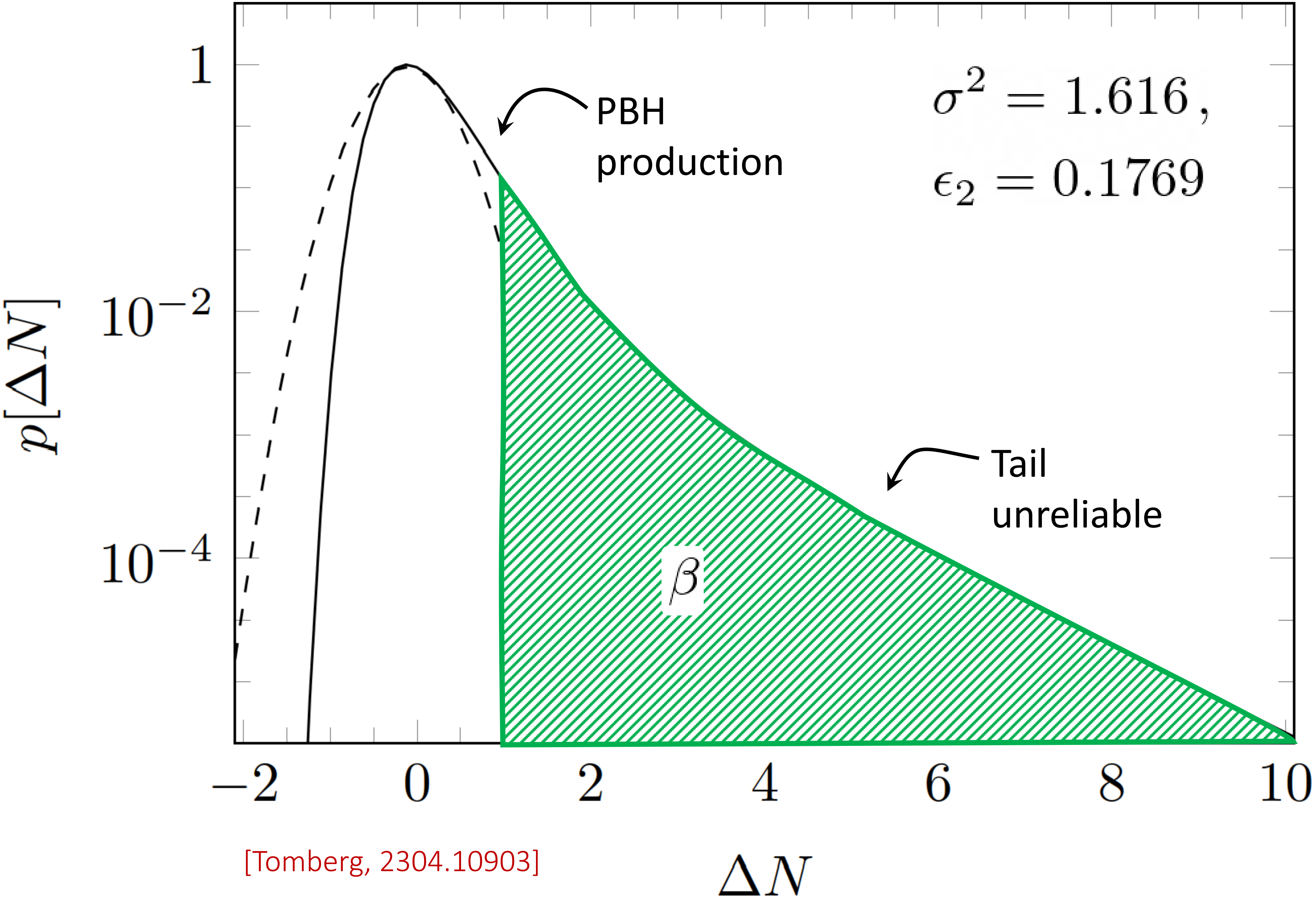
$$X = \frac{2}{\epsilon_2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N}\right)$$

$$p(\Delta N) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{2}{\sigma^2 \epsilon_2^2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N}\right)^2 - \frac{\epsilon_2}{2}\Delta N\right]$$

$\Delta N = \mathcal{R}$



[Tomberg, 2304.10903]



[Tomberg, 2304.10903]

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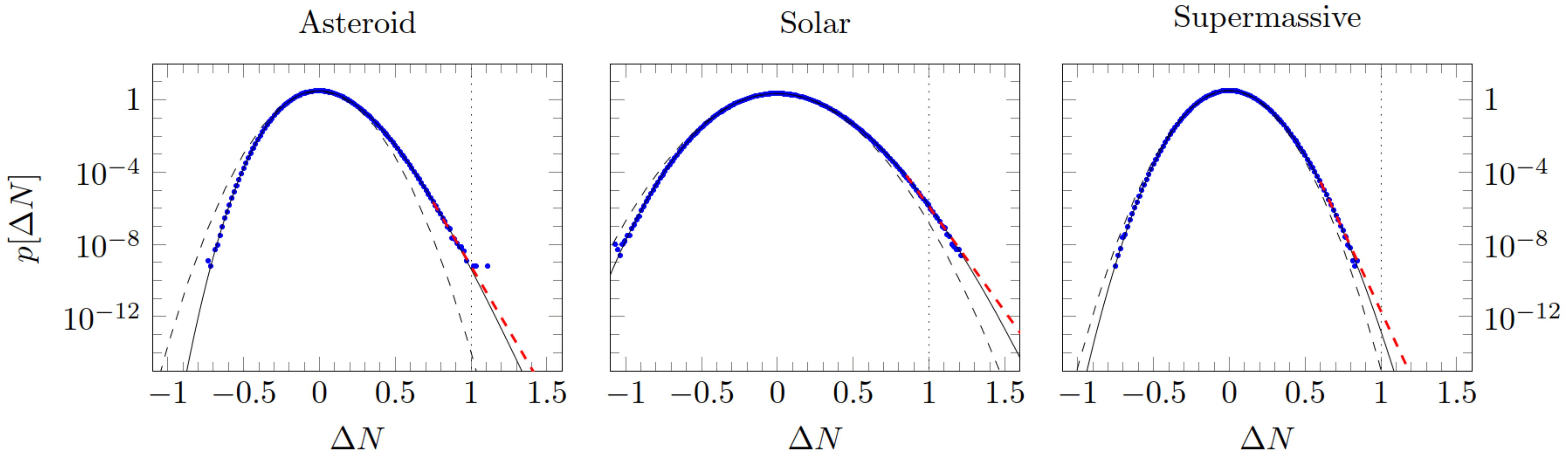

Black = Constant-roll approximation

Dashed = Gaussian fit

Blue = numerical computation

Red = numerical extrapolation

Comparison to non-stochastic ΔN

[Cai et al, 1712.09998]
[Biagetti et al, 2105.07810]
[Pi et al, 2211.13932]
…

Same result without stochasticity:

- Compute "total field perturbation" $\Delta\phi$
- Convert to ΔN using classical background eom

"One initial kick"

Works in constant-roll due to linearity of background eom

Conclusions

Stochastic inflation introduces non-Gaussian corrections to PBH statistics

Constant-roll inflation is at the heart of PBH inflation models

$$p(\mathcal{R}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{2}{\sigma^2 \epsilon_2^2}\left(1 - e^{-\frac{\epsilon_2}{2}\mathcal{R}}\right)^2 - \frac{\epsilon_2}{2}\mathcal{R}\right]$$