Revisiting warm inflation

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Based on 2304.05978 with G. Ballesteros, and M. Pierre

- Cosmology from home. July 2023 -

Outline

1. **Basics**
2. **Analytical insight**: formal issues impact phenomenology
3. **Phenomenology**: new method and results

1. Warm inflation basics: background

Main idea: **dissipation** from the inflaton ϕ into a **thermalised** bath with $\rho_r \propto T^4$.

$$\ddot{\phi} + (3H + \Gamma) + V_{,\phi} = 0\,,$$
$$\dot{\rho}_r + 4H\rho_r = \Gamma\dot{\phi}^2\,.$$

- The effective dissipative coefficient Γ encloses information about (indirect) coupling of inflaton to light degrees of freedom (see review in [*Kamali, Motaharfar, Ramos '23*]).
- Useful notation: $Q = \frac{\Gamma}{3H}$.
  $Q = 0$: cold inflation. $Q \ll 1$: weak dissipation; $Q \gg 1$: strong dissipation.

1. Warm inflation basics: perturbations

[*Bastero-Gil, Berera, Moss, Ramos '14*]

- **Continuity** equations for $T^{0\nu}_{(\phi)}$ and $T^{0\nu}_{(r)}$:

$$\delta\ddot{\phi}_{\boldsymbol{k}} + (3H+\Gamma)\delta\dot{\phi}_{\boldsymbol{k}} + \left(\frac{k^2}{a^2} + V_{\phi\phi} + \dot{\phi}\Gamma_{\phi}\right)\delta\phi_{\boldsymbol{k}} + \Gamma_T\frac{\dot{\phi}T}{4\rho_r}\delta\rho_{r,\boldsymbol{k}} - 4\dot{\psi}_{\boldsymbol{k}}\dot{\phi} + (2V_{\phi}+\Gamma\dot{\phi})\psi_{\boldsymbol{k}} = \sqrt{\frac{2\Gamma T}{a^3}}\,\xi_{\boldsymbol{k}}(t)\,,$$

$$\delta\dot{\rho}_{r,\boldsymbol{k}} + \left(4H - \Gamma_T\frac{\dot{\phi}^2 T}{4\rho_r}\right)\delta\rho_{r,\boldsymbol{k}} - \frac{k^2}{a^2}\delta q_{r,\boldsymbol{k}} + \Gamma\dot{\phi}^2\psi_{\boldsymbol{k}} - 4\rho_r\dot{\psi}_{\boldsymbol{k}} - (\Gamma_{\phi}\delta\phi_{\boldsymbol{k}} - 2\Gamma\delta\dot{\phi}_{\boldsymbol{k}})\dot{\phi} = \underbrace{-\sqrt{\frac{2\Gamma T}{a^3}}\dot{\phi}\,\xi_{\boldsymbol{k}}(t)}_{\text{FD thm.}}\,,$$

  where $\langle\xi_{\boldsymbol{k}}(t)\rangle = 0, \quad \langle\xi_{\boldsymbol{k}}(t)\xi_{\boldsymbol{k}'}(t')\rangle = \delta(t-t')\delta(\boldsymbol{k}+\boldsymbol{k}')$ (white noise).

- **Einstein** equation: $\dot{\psi}_{\boldsymbol{k}} + H\psi_{\boldsymbol{k}} = -\frac{1}{2M_P^2}\left(\delta q_{r,\boldsymbol{k}} - \dot{\phi}\,\delta\phi_{\boldsymbol{k}}\right)$,

- **Constraint** for $\delta q_{r,\boldsymbol{k}}$: $\left(2M_P^2\frac{k^2}{a^2} - \dot{\phi}^2\right)\psi_{\boldsymbol{k}} + \delta\rho_{r,\boldsymbol{k}} + \dot{\phi}\,\delta\dot{\phi}_{\boldsymbol{k}} + (V_{\phi} + 3H\dot{\phi})\delta\phi_{\boldsymbol{k}} - 3H\delta q_{r,\boldsymbol{k}} = 0.$

Complete system of equations which allows to compute $\mathcal{R}_{\boldsymbol{k}} = \frac{H}{\rho+p}\left(\delta q_{r,\boldsymbol{k}} - \dot{\phi}\,\delta\phi_{\boldsymbol{k}}\right) - \psi_{\boldsymbol{k}}$.

2. Analytical insight: main equation

Several approximations:

- Consider only equation for $\delta\phi_{\boldsymbol{k}}$ (main contribution to $\mathcal{R}_{\boldsymbol{k}}$): $\mathcal{R}_{\boldsymbol{k}} \approx -\frac{H}{\rho+p}\dot{\phi}\,\delta\phi_{\boldsymbol{k}}$
- Neglect slow-roll suppressed terms.

$$\delta\ddot{\phi}_{\boldsymbol{k}} + (3H + \Gamma)\delta\dot{\phi}_{\boldsymbol{k}} + \frac{k^2}{a^2}\delta\phi_{\boldsymbol{k}} + \Gamma_T \frac{\dot{\phi}T}{4\rho_r}\delta\rho_{r,\boldsymbol{k}} = \sqrt{\frac{2\Gamma T}{a^3}}\,\xi_{\boldsymbol{k}}(t)$$

Three departures from cold inflation:

1. Dissipative term
2. Coupling to radiation perturbations
3. Stochastic (thermal) source (fluctuation-dissipation theorem)

2. Analytical insight: quantization

$$\delta\ddot{\phi}_{\boldsymbol{k}} + (3H + \Gamma)\delta\dot{\phi}_{\boldsymbol{k}} + \frac{k^2}{a^2}\delta\phi_{\boldsymbol{k}} + \Gamma_T \frac{\dot{\phi}T}{4\rho_r}\delta\rho_{r,\boldsymbol{k}} = \sqrt{\frac{2\Gamma T}{a^3}}\,\xi_{\boldsymbol{k}}(t)$$

1. Change time variable to conformal time $a\,d\eta = dt$
2. Field redefinition $\chi_{\boldsymbol{k}} = \delta\phi_{\boldsymbol{k}}\, a^{1+3Q/2}$, (c.f. cold inflation)

We end up with (c.f. Mukhanov-Sasaki equation) [*Ballesteros, APR, Pierre '23*]

$$\overbrace{\frac{d^2\chi_{\boldsymbol{k}}}{d\eta^2} + \omega_k^2(\eta)\chi_{\boldsymbol{k}}}^{\text{free field}} + \overbrace{\Gamma_T \frac{\dot{\phi}T}{4\rho_r}\delta\rho_{r,\boldsymbol{k}}}^{\text{interaction term}} = \overbrace{S(\eta)\xi_{\boldsymbol{k}}(\eta)}^{\text{classical source}},$$

$$\text{with} \quad \omega_k^2 = k^2 + \frac{1 - 9(1+Q)^2}{4\eta^2}, \qquad S \propto \sqrt{\Gamma T}$$

2. Analytical insight: quantization

In the weak dissipation limit ($Q \ll 1$), the interaction term is negligible. The field is quantized as [*Mukhanov, Winitzki '07*]

$$\hat{\chi}_{\boldsymbol{k}} = \underbrace{\chi_k^{(h)} \hat{a}_{\boldsymbol{k}} + \text{h.c.}}_{\text{quantum component}} + \underbrace{\mathbb{I} \int^{\eta} d\eta' \, G_k^{(\text{ret})}(\eta, \eta') S_{\boldsymbol{k}}(\eta') \xi_{\boldsymbol{k}}(\eta')}_{\text{classical thermal (stochastic) component}}$$

This allows to compute the dimensionless power spectrum, defined as

$$\langle \langle 0 | \hat{\chi}(\boldsymbol{x}, \eta) \, \hat{\chi}(\boldsymbol{x}, \eta) | 0 \rangle \rangle = \int d(\log k) \, \langle \mathcal{P}_{\chi}(k, \eta) \rangle \, ,$$

where $\langle 0 | \cdot | 0 \rangle$ is a **quantum** expectation value (involves operators $\hat{a}_{\boldsymbol{k}}, \hat{a}^{\dagger}_{\boldsymbol{k}}, \mathbb{I}$) and $\langle \cdot \rangle$ is an **average over realizations** of the noise (i.e. ξ).

2. Analytical insight: averaged power spectrum

The averaged power spectrum naturally decomposes *[Ballesteros, APR, Pierre '23]*

$$\boxed{\langle \mathcal{P}_\chi(k,\eta)\rangle = \mathcal{P}_\chi^{(h)}(k,\eta) + \langle \mathcal{P}_\chi^{(i)}(k,\eta)\rangle \,.}$$

Recall: $\mathcal{P}_\chi(k,\eta) \propto \mathcal{P}_{\delta\phi}(k,\eta) \propto \mathcal{P}_{\mathcal{R}}(k,\eta)$ (aprox.)

$$\mathcal{P}_\chi^{(h)}(k,\eta) = \boxed{\frac{k^3}{2\pi^2}|\chi_k^{(h)}|^2} \quad \text{(c.f. cold inflation)}$$

$$\langle \mathcal{P}_\chi^{(i)}(k,\eta)\rangle = \int^\eta d\eta' \left[G_k^{(\mathrm{ret})}(\eta,\eta')\right]^2 S_{\boldsymbol{k}}^2(\eta') \overset{(1)}{\propto} \boxed{\Gamma_* T_* \int^\eta d\eta' \left[G_k^{(\mathrm{ret})}(\eta,\eta')\right]^2}$$

[1]Recall: $\langle \xi_{\boldsymbol{k}}(t)\xi_{\boldsymbol{k'}}(t')\rangle = \delta(t-t')\delta(\boldsymbol{k}+\boldsymbol{k'})$

2. Analytical insight: occupation number of $\delta\phi_{\boldsymbol{k}}$

If inflaton perturbations are in equilibrium with the thermal bath,

$$\mathcal{P}^{(h)}_{\delta\phi}(k,\eta) = \frac{k^3}{2\pi^2}|\delta\phi^{(h)}_k|^2 \underbrace{(1+2n_{\rm BE})}_{\text{thermal correction}} \quad , \quad n_{\rm BE} = \frac{1}{e^{\frac{H}{T}-1}-1}$$

Thermal correction arises from $\langle \hat{a}_{\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k}'} + \text{h.c.}\rangle_{\text{th.eq.}}$, see e.g. [*Ramos, da Silva '13*].

- Some papers consider both possibilities on equal footing: [*Bastero-Gil, Berera, Moss, Ramos '14*], [*Bastero-Gil, Berera, Ramos, Rosa '16*]...
- In many WI models, the inflaton only couples **indirectly** to the thermal bath (exceptions: e.g. axion inflation), no guarantee of thermalisation.
- The issue is **model dependent**. Heavy impact on predictions because $n_{\rm BE} \gg 1$ if $T \gg H$.

2. Analytical insight: initial conditions of $\delta\phi_{\boldsymbol{k}}$

Assume inflaton perturbations were in Bunch-Davies in the far past (i.e. inflation was cold at some point). If transition to warm inflation occurred at some scale in the far past $z_0 = k/(a_0 H) \gg 1$, [*Nacir, Porto, Senatore, Zaldarriaga '12*], [*Ballesteros, APR, Pierre '23*]

- For $Q \gg 1$, $\mathcal{P}^{(h)}_{\delta\phi}(k) \sim \frac{{z_0}^3}{Q} \underbrace{\left(\frac{2\nu}{z_0 e}\right)}_{\ll 1}^{3Q}$ **(exponential suppression with Q)**
- For $Q \ll 1$, $\mathcal{P}^{(h)}_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2 + \mathcal{O}(\log z_0) Q$ **(recovers cold limit)**

Two consequences:

- **At large Q, the thermal contribution dominates**
- For $Q \lesssim 1$, $\mathcal{P}_{\delta\phi} = \mathcal{P}^{(h)}_{\delta\phi} + \langle \mathcal{P}^{(i)}_{\delta\phi} \rangle < \left(\frac{H}{2\pi}\right)^2 + \langle \mathcal{P}^{(i)}_{\delta\phi} \rangle$ (especially if thermal correction applies). C.f. [*Bartrum, Bastero-Gil, Berera, Cerezo, Ramos, Rosa '14*] and refs. therein, also discussion in [*Ballesteros, APR, Pierre '23*].

2. Analytical insight: illustration

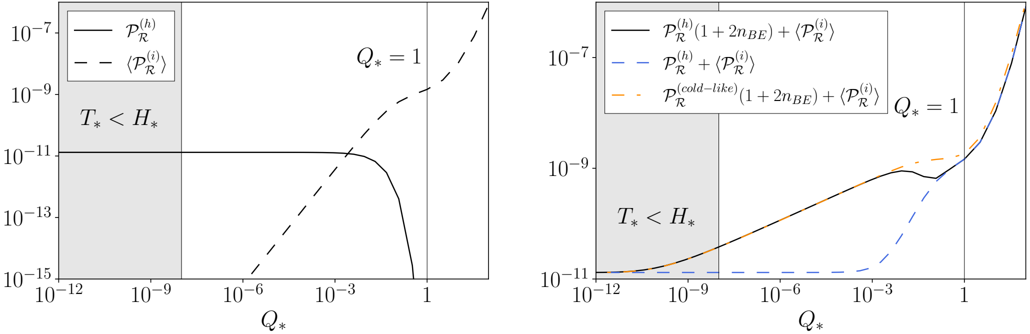

Figure: $V(\phi) = \lambda\,\phi/4$, $\lambda = 10^{-15}$, $\Gamma(T) = C\,T$. Assuming **Bunch-Davies initial conditions** 5 e-folds before horizon crossing.

3. Phenomenology: reminder of equations

C.f. slide 4

- **Continuity** equations for $T^{0\nu}_{(\phi)}$ and $T^{0\nu}_{(r)}$:

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Complete system of equations which allows to compute $\mathcal{R}_{\boldsymbol{k}} = \frac{H}{\rho+p}\left(\delta q_{r,\boldsymbol{k}} - \dot{\phi}\,\delta\phi_{\boldsymbol{k}}\right) - \psi_{\boldsymbol{k}}$.

3. Phenomenology: numerical procedure

In order to compute $\langle \mathcal{P}_{\mathcal{R}}^{(i)} \rangle$, previous approaches were based on Montecarlo averages. Instead, we propose a **Fokker-Planck approach** (c.f. stochastic inflation).
[*Ballesteros, García, APR, Pierre, Rey '22*].

- Recast system of SDEs for $\delta\phi_{\boldsymbol{k}}$, $\delta\rho_{r,\boldsymbol{k}}$, $\psi_{\boldsymbol{k}}$ into system of ODEs for $\langle |\delta\phi_{\boldsymbol{k}}|^2 \rangle$, $\langle |\delta\rho_{r,\boldsymbol{k}}|^2 \rangle$, $\langle |\psi_{\boldsymbol{k}}|^2 \rangle$, . . . , $\langle \delta\phi^*_{\boldsymbol{k}} \delta\rho_{r,\boldsymbol{k}} \rangle$, . . .
- Solve the system **once** and recast the result into $\langle \mathcal{P}_{\mathcal{R}}^{(i)} \rangle$
- Fast, no statistical error, accuracy only limited by numerical precision

3. Phenomenology: CMB constraints

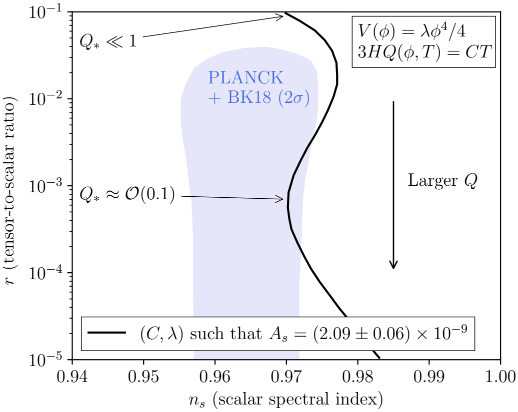


Figure: Behaviour of isocurves of $\langle \mathcal{P}_\mathcal{R}(k) \rangle = \mathcal{P}_\mathcal{R}^{(h)}(k) + \langle \mathcal{P}_\mathcal{R}^{(i)}(k) \rangle$

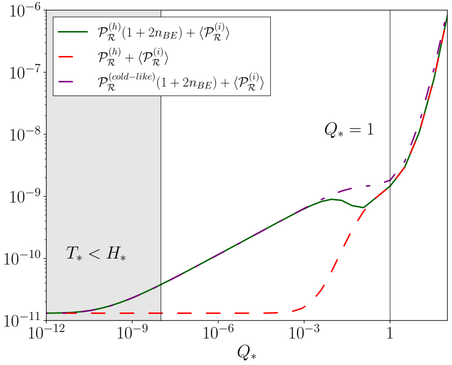
	ϕ^6	ϕ^4	ϕ^2
T	Yes	Yes	No
T^3	No	Yes	No
T^3/ϕ^2	No	No	No

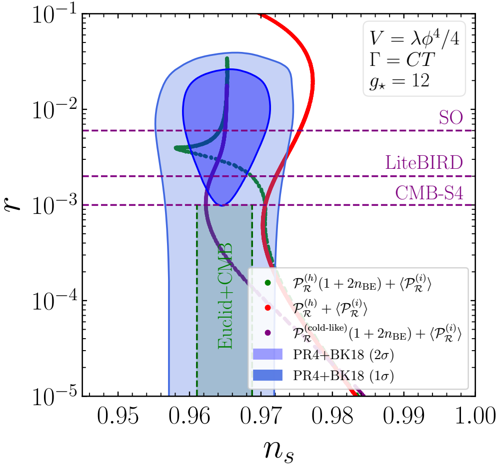
Table: Rows: $\Gamma(\phi, T)$. Cols.: $V(\phi)$

[*Ballesteros, APR, Pierre '23*]

3. Phenomenology: CMB constraints

[*Ballesteros, APR, Pierre '23*], c.f. [*Bartrum, Bastero-Gil, Berera, Cerezo, Ramos, Rosa '14*]





Conclusions

- Warm inflation leads to a stochastic source for the perturbations.
- The averaged power spectrum has two components:

$$\langle \mathcal{P}_{\mathcal{R}}(k) \rangle = \mathcal{P}_{\mathcal{R}}^{(h)}(k) + \langle \mathcal{P}_{\mathcal{R}}^{(i)}(k) \rangle \,,$$

  $\langle \mathcal{P}_{\mathcal{R}}^{(h)} \rangle$ *depends* on initial conditions, has a quantum origin and is **suppressed by dissipation** (smaller for larger Q). **Recovers cold limit. Thermal correction?** $\langle \mathcal{P}_{\mathcal{R}}^{(i)} \rangle$ *does not* depend on the initial conditions and is **due purely to the thermal noise** (larger for larger Q). **Disappears in the cold limit**.
- $\langle \mathcal{P}_{\mathcal{R}}^{(i)} \rangle$ can be efficiently computed with a **Fokker-Planck** approach.
- *Some* monomial warm inflation spectra are reconciled with CMB constraints.